General two-order-parameter Ginzburg-Landau model with quadratic and quartic interactions

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The Ginzburg-Landau model with two-order parameters appears in many condensed-matter problems. However, even for scalar order parameters, the most general U(1)-symmetric Landau potential with all quadratic and quartic terms contains 13 independent coefficients and cannot be minimized with straightforward algebra. Here, we develop a geometric approach that circumvents this computational difficulty and allows one to study properties of the model without knowing the exact position of the minimum. In particular, we find the number of minima of the potential, classify explicit symmetries possible in this model, establish conditions when and how these symmetries are spontaneously broken, and explicitly describe the phase diagram.

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I. INTRODUCTION

The Landau theory [1,2] offers a remarkably economic description of phase transitions associated with symmetry breaking. This breaking is described by an order parameter ψ: the high-symmetry phase corresponds to $\psi=0$, while the low-symmetry phase is described by $\psi\neq 0$. Very often, the order parameter can be directly related to physically observable quantities, such as, for example, distortion of the crystal lattice or spontaneous magnetization. The local version of the Landau theory with a coordinate-dependent order parameter, known as Ginzburg-Landau (GL) theory, is the basis of the phenomenological theory of superconductivity [3]. For a variety of applications of the Landau theory to various condensed-matter problems, see, e.g., [4] and references therein.

In order to find if a given system is in its high- or low-symmetry phase, one constructs a Landau potential that depends on the order parameter, and then finds its minimum. For a complex order parameter, its classic form [1,2] is

$$V(\psi) = -a|\psi|^2 + \frac{b}{2}|\psi|^4 + o(|\psi|^4). \tag{1}$$

Near the phase transition, the higher-order terms $o(|\psi|^4)$ are usually assumed to be negligible. The values of the coefficients a and b and their dependence on temperature, pressure, etc. can be either calculated from a microscopic theory, if it is available, or considered as free parameters in a phenomenological approach. The phase transition associated with the symmetry breaking takes place when an initially negative a becomes positive, and the minimum of the potential (1) shifts from zero to

$$\langle\psi\rangle = \sqrt{\frac{a}{b}}e^{i\alpha}, \tag{2}$$

with an arbitrary phase α.

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Many systems are known in which two competing order parameters (OP) coexist. Among them are the general $O(m)\oplus O(n)$-symmetric models [5]; the models with two interacting N-vector OPs with $O(N)$ symmetry [6]; spin-density waves in cuprates [7]; competition between antiferromagnetism and superconductivity [8]; ^{4}He with its interplay of crystalline and superfluid ordering [9]; multicomponent [10]; nonconventional two-dimensional [11]; spin-triplet p-wave [12]; and two-gap [13,14] superconductivity, with its application to magnetism in neutron stars [15]; two-band superfluidity [16]; and even mechanisms of electroweak symmetry breaking beyond the Standard Model such as the two-Higgs-doublet model (2HDM) [17].

To describe such a situation within GL theory, one constructs a Landau potential similar to Eq. (1), which depends on two order parameters, ψ_1 and ψ_2. With scalar order parameters, it can be written generically as

$$V(\psi_1,\psi_2) = -a_{ij}(\psi_i^*\psi_j) + \frac{1}{2}b_{ijkl}(\psi_i^*\psi_j)(\psi_k^*\psi_l), \quad i,j,k,l=1,2. \tag{3}$$

The coefficients of this potential can be considered independent, although in each particular application they might obey specific relations. One thus arrives at the general two-order-parameter (2OP) GL model with quadratic and quartic terms.

Once Landau potential (3) is written, the next step is to find its minimum, i.e., to solve the static homogeneous Ginzburg-Landau equations. A rather surprising fact is that these equations cannot be solved with straightforward algebra. Differentiating the Landau potential with respect to ψ_i leads to a system of coupled algebraic equations of third order, whose total degree of algebraic complexity is six, which makes it impossible to solve in the general case.

In this paper, we argue that despite this computational problem, there remains something that one can learn about the most general two-order-parameter model in the mean-field approximation: its *phase diagram*. As we will show below, it is possible to classify all the phases according to the symmetries of the model and properties of the ground state.

This idea is not new. In fact, there exists an extensive literature dating back to the 1980s on minimization of G-invariant potentials with several OPs (G being a group of transformations of OPs); see, e.g., [4,18,19] and references therein. These works exploit the fact that the problem becomes simpler when reformulated in the orbit space instead of the space of order parameters themselves [20]. This orbit space is naturally sliced into several strata, which are linked to the allowed phases of the model. To describe them, one constructs the ring of G-invariant polynomials of the order parameters and finds the minimal integrity basis (MIB) of this ring. Different strata (i.e., different phases) correspond to some particular relations among the MIB polynomials. This general method has been applied to classification of the phases of several relevant physical systems, for example to p-wave superfluidity of ^{3}He [21], D-wave condensates [22], 2HDM [23,24], and to general Landau models with multicomponent and even multidimensional order parameters [25,26].

In light of this activity, it is somewhat surprising that the most general GL model with two complex order parameters and with the most general potential in the form of Eq. (3) has never been studied in complete detail. Here we fill this gap showing that in this case the analysis can be pushed much farther than in the general situation, with important physical consequences.

The approach presented here is based on the reparametrization symmetry of the model, which allows one to establish the Minkowski-space structure of the orbit space. The minimization problem admits a transparent geometric interpretation, which leads to several theorems concerning the properties of the global minimum. Specific application of this approach to the 2HDM was given in [28,29]. Here we analyze the case of two local order parameters $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$ in the general context, which can be relevant also for many condensed-matter problems.

Geometric approach versus minimal integrity basis method

Let us stress from the very start the essential differences between the geometric analysis of the present paper and the standard approach based on the minimal integrity basis (MIB) technique.

The first difference lies in the scope of these two approaches. The MIB leads to interesting results in the cases in which the potential is invariant under a nontrivial group G of transformation of the order parameters. The larger G is, the richer is the spectrum of possible patterns of its spontaneous violation. In particular, MIB methods have no role if G is the trivial group.

In a typical situation one takes a highly symmetric G-invariant potential constructed from powers of several multidimensional order parameters up to a certain degree, builds various invariants, finds the ones that form MIB, and classifies the possible phases according to relations among these invariants. This approach is rather general in a sense that it can be applied, in principle, to any number and any dimensions of order parameters. However, because of G invariance, the potentials usually contain very few terms.

The geometric approach presented here is limited to the particular case of two complex scalar or vector order parameters, and to the fourth-degree potentials. However, within these restrictions, we manage to work out the *most general* model with all possible types of the OP interactions. The only symmetry that we impose is the U(1) symmetry of the free-energy density, which is a reasonable choice from the physical point of view. In this aspect, our analysis is more general than the MIB approach: we just take two OPs, construct the free energy density in its full complexity, and study *everything* that can ever happen in this model. That is, we analyze all possible symmetry groups G and all possible patterns of symmetry breaking.

The second difference concerns the procedures and the results of these two approaches. In a situation when several phases are possible, one wants to know which phase corresponds to the ground state of the model (i.e., which phase is stable). In the usual MIB method, one can do nothing but explicitly solve the algebraic equations and check the minimum conditions. This can be done only if the equations are simple enough, which in turns happen when the free-energy density is simple. Thus, only sufficiently symmetric potentials are fully tractable with the MIB method. Examples cited in [24,26] are precisely of this type.

In terminology suggested by [27], one should distinguish between the *angular problem* (classifying all the allowed phases) and the *radial problem* (actually finding the position of the absolute minimum of a given potential). MIB methods allow one to solve the angular but not the radial problem.

In the case of the most general 2OP GL model, with its large number of free parameters, this algebra cannot be worked out explicitly. One ends up with a general algebraic equation of sixth order, which one cannot solve analytically. Thus, one is unable to solve the problem of minimization of the potential using only the MIB formalism.

The strongest point of the present geometric approach is that we *avoid* solving these equations and nevertheless we rigorously prove several statements about the ground state of the model. In other words, we study the properties of the absolute minimum *without* solving the radial problem. This is especially useful for the case of the *smallest* possible group G, for which the MIB technique becomes redundant. Thus, the geometric approach presented here is neither a particular case nor an improvement of the MIB method, but is complementary to it.

The structure of the paper is the following. In Sec. II, we introduce the formalism and derive a very compact expression for the free-energy functional. The extrema of the Landau potential cannot be found with straightforward algebra, so in Sec. III we develop geometric tools that allow us to find the number of extrema and minima of the potential. Section IV is devoted to the special case of a potential stable in a weak sense. Then, in Sec. V we give full classification of explicit symmetries of the model and derive conditions when and how these symmetries are spontaneously broken. All this allows us to describe in Sec. VI the phase diagrams of the model, listing the phases according to the number of minima and symmetries. Here, we also discuss phase transitions and argue that critical properties, too, can be calculated in geometric terms. Sec. VII contains analysis of several simple

cases, which provide illustration of the general approach. In the short Sec. VIII, we outline conditions when solitons appear in this model. In Sec. IX, we outline characteristic features of the general GL model with two complex N-vector order parameters, and finally in Sec. X we draw our conclusions. Appendixes provide some mathematical details and derivations.

II. FORMALISM

In the main part of the paper, we will assume that $\psi_i(\vec{r})$ are just complex numbers; modifications in the case of more complicated OPs will be discussed in Sec. IX. Throughout the paper, we also assume that the absolute values of $|\psi_i|$ are not bounded from above.

Let us consider the free-energy density in the most general globally U(1)-invariant 2OP GL model containing all possible quadratic and quartic terms in the potential,

$$F = K + V_2 + V_4. \tag{4}$$

It is a sum of the gradient term K,

$$K = \kappa_1|\vec{D}\psi_1|^2 + \kappa_2|\vec{D}\psi_2|^2 + \kappa_3(\vec{D}\psi_1)^*(\vec{D}\psi_2) + \kappa_3^*(\vec{D}\psi_2)^*(\vec{D}\psi_1), \tag{5}$$

where $\vec{D}$ is either $\vec{\nabla}$ or the covariant derivative, and the Landau potential

$$V_2 = -a_1|\psi_1|^2 - a_2|\psi_2|^2 - a_3(\psi_1^*\psi_2) - a_3^*(\psi_2^*\psi_1), \tag{6}$$

$$V_4 = \frac{b_1}{2}|\psi_1|^4 + \frac{b_2}{2}|\psi_2|^4 + b_3|\psi_1|^2|\psi_2|^2 + \left[\frac{b_4}{2}(\psi_1^*\psi_2) + b_5|\psi_1|^2 + b_6|\psi_2|^2\right](\psi_1^*\psi_2) + \text{c.c.}$$

Free-energy density (4) contains $4+4+9=17$ free parameters: real $\kappa_1,\kappa_2,a_1,a_2,b_1,b_2,b_3$ and complex κ_3,a_3,b_4,b_5,b_6.

By construction, the free energy remains invariant under the U(1) group of simultaneous multiplication of ψ_1 and ψ_2 by the same global phase factor. We do not consider terms that violate this symmetry, such as $\psi_1^2+(\psi_1^2)^*$.

Note that potential (6) contains quartic terms such as $|\psi_1|^2(\psi_1^*\psi_2)$ that mix ψ_1 and ψ_2, which are usually absent in many particular applications of the 2OP GL model. However, in certain cases such terms appear, as it happens in the dirty limit of a two-gap superconductor; see, e.g., [14].

We stress that in our approach, it is essential that we include all possible terms from the very beginning.

A. Reparametrization symmetry

From the physical point of view, the order parameters ψ_1 and ψ_2 can be of the same (as in two-gap superconductors) or of different nature (as in the case of superfluid and crystalline ordering interplay). However, one can always make OPs dimensionless, and once the free-energy density (4) is constructed and the problem of its minimization is posed, the physical nature of the OPs becomes irrelevant.

One can then view OPs ψ_1 and ψ_2 as components of a single complex 2-vector Φ,

$$\Phi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

and consider transformations that mix ψ_1 and ψ_2. These are assumed to be local transformations, i.e., they mix $\psi_i(\vec{r})$ taken at the same point $\vec{r}$.

We start with the observation that the most general free-energy density (4) retains its generic form under any regular linear transformation between ψ_1 and ψ_2. In other words, any transformation from the general linear group $\mathrm{GL}(2,C)$

$$\Phi \to \Phi' = T\Phi, \quad T \in \mathrm{GL}(2,C) \tag{7}$$

again leads to Eq. (4) but with reparametrized coefficients,

$$\{\kappa_i,a_i,b_i\} \to \{\kappa_i',a_i',b_i'\} = \tau(\kappa_i,a_i,b_i). \tag{8}$$

The explicit link between T and τ will be given below.

Since any $T \in \mathrm{GL}(2,C)$ is invertible, so is τ. Therefore, if Eq. (7) is accompanied by the transform τ^{-1} of the coefficients, then one arrives at *exactly the same* expression for the free energy as before.

If one considers the free energy only, then the physical observables, such as the depth of the Landau potential at the minimum and the eigenvalues of the second derivative matrix of the potential (the Hessian), can be expressed in terms of the coefficients $\{\kappa_i,a_i,b_i\}$ only. Therefore, the models $(\Phi,\{\kappa_i,a_i,b_i\})$ and $(\Phi',\{\kappa_i',a_i',b_i'\})$ have the same sets of observables. In other words, reparametrization transformations do not change the physical content of a given model; they only affect the way we look at it. Thus, we have a *reparametrization freedom* in this problem, with the reparametrization group $\mathrm{GL}(2,C)$.

The general linear group $\mathrm{GL}(2,C)$ is an eight-dimensional Lie group. It can be written as

$$\mathrm{GL}(2,C) = \mathbb{C}^* \times \mathrm{SL}(2,C), \tag{9}$$

where $\mathbb{C}^*$ is the group of all multiplications of Φ by a nonzero complex number and $\mathrm{SL}(2,C)$ is the special linear group. Due to the U(1) invariance of the free energy, multiplication of Φ by an overall phase factor induces the identity transformation of the coefficients, while the seven-dimensional factor group $\mathrm{GL}(2,C)/\mathrm{U}(1)$ induces nontrivial transformations τ. Thus, the 17-dimensional space of coefficients (i.e., the space of all possible 2OP GL models) becomes sliced into seven-dimensional regions of essentially identical models linked by all possible τ. The space of distinct physical situations is described by the corresponding 10-dimensional factor space.

B. Orbit space

Let us now introduce the four-vector $r^\mu=(r_0,r_i)=(\Phi^\dagger\sigma^\mu\Phi)$ with components

$$r_0 = (\Phi^\dagger \Phi) = |\psi_1|^2 + |\psi_2|^2, \quad r_i = (\Phi^\dagger \sigma_i \Phi) = \begin{pmatrix} 2\,\mathrm{Re}(\psi_1^* \psi_2) \\ 2\,\mathrm{Im}(\psi_1^* \psi_2) \\ |\psi_1|^2 - |\psi_2|^2 \end{pmatrix}. \tag{10}$$

Here, index $\mu = 0,1,2,3$ refers to the components in the internal space and has no relation to the space-time. Multiplying ψ_i by a common phase factor does not change r^μ, so each r^μ uniquely parametrizes a single U(1) orbit in the ψ_i space. The U(1)-invariant free energy (4) can also be defined in this (1+3)-dimensional *orbit space*.

The $SL(2,C) \subset GL(2,C)$ group of transformations of Φ induces the proper Lorentz group SO(1, 3) of transformations of r^μ. This group includes 3D rotations of the vector r_i as well as "boosts" that mix r_0 and r_i, so the orbit space gets naturally equipped with the *Minkowski space structure* with metric diag(1,−1,−1,−1). We stress again that the words "Minkowski space" and "Lorentz group" always refer to the internal space, not to the usual space-time.

Since the order parameters ψ_1 and ψ_2 are just complex numbers, direct calculation shows that

$$r^\mu r_\mu \equiv r_0^2 - r_i^2 = 0. \tag{11}$$

Then, since $r_0 > 0$ and since the values of r^μ are not restricted from above, the orbit space of the 2OP GL model is given by the forward lightcone LC^+ in the Minkowski space. As should be expected, the reparametrization group in the orbit space, SO(1, 3), leaves the orbit space invariant.

Analogously to r^μ, one can also introduce

$$\rho^\mu \equiv (\vec{D}\Phi)^\dagger \sigma^\mu (\vec{D}\Phi). \tag{12}$$

Obviously, the reparametrization transformation laws of ρ^μ are the same as for r^μ.

All this allows us to rewrite the free energy (4) in a very compact form,

$$F = K_\mu \rho^\mu - A_\mu r^\mu + \frac{1}{2} B_{\mu\nu} r^\mu r^\nu, \tag{13}$$

with

$$K^\mu = \frac{1}{2}(\kappa_1 + \kappa_2, -2\,\mathrm{Re}\,\kappa_3, 2\,\mathrm{Im}\,\kappa_3, -\kappa_1 + \kappa_2),$$

$$A^\mu = \frac{1}{2}(a_1 + a_2, -2\,\mathrm{Re}\,a_3, 2\,\mathrm{Im}\,a_3, -a_1 + a_2),$$

$$B^{\mu\nu} = \frac{1}{2}\begin{pmatrix} \frac{b_1+b_2}{2} + b_3 & -\mathrm{Re}(b_5+b_6) & \mathrm{Im}(b_5+b_6) & -\frac{b_1-b_2}{2} \\ -\mathrm{Re}(b_5+b_6) & \mathrm{Re}\,b_4 & -\mathrm{Im}\,b_4 & \mathrm{Re}(b_5-b_6) \\ \mathrm{Im}(b_5+b_6) & -\mathrm{Im}\,b_4 & -\mathrm{Re}\,b_4 & -\mathrm{Im}(b_5-b_6) \\ -\frac{b_1-b_2}{2} & \mathrm{Re}(b_5-b_6) & -\mathrm{Im}(b_5-b_6) & \frac{b_1+b_2}{2} - b_3 \end{pmatrix}. \tag{14}$$

Note that due to Eq. (11), definition of the matrix $B^{\mu\nu}$ has one degree of freedom, since $B^{\mu\nu}$ and $\widetilde{B^{\mu\nu}} = B^{\mu\nu} - C g^{\mu\nu}$ with any C are equivalent.

The quantities K^μ, A^μ, and $B^{\mu\nu}$ transform as 4-vectors and a 4-tensor, respectively. This provides the explicit link between transformations T and τ mentioned in Sec. II A. For convenience, we collect in Appendix A some basic facts concerning the manipulation of $B^{\mu\nu}$.

C. Properties of the coefficients

General physical requirements place restrictions on possible K_μ and $B_{\mu\nu}$.

First, the requirement that very large wave-vector oscillations must increase, not decrease, the free energy implies that K_μ lies inside the future lightcone: $K_0 > 0$, $K_\mu K^\mu > 0$. This condition remains true under an arbitrary SO(1, 3) transformation.

Second, we require that the potential is bounded from below in the entire ψ_i space. Since the potential is built out of quadratic and quartic terms, $V = V_2 + V_4$, this can be achieved in two cases (here we coin the terminology of [30], where the stability of the Higgs potential in 2HDM was analyzed): (i) the potential is stable in a *strong* sense, if V_4 increases in all directions in the ψ_i space; and (ii) the potential is stable in a *weak* sense, if V_4 is nondecreasing in all directions in the ψ_i space, and V_2 increases along the flat directions of V_4.

Let us focus on the case of the potentials stable in a strong sense; the case of the potential stable in a weak sense will be considered in Sec. IV. The requirement that V_4 is positive definite in the entire ψ_i space means that the quadratic form $B_{\mu\nu} r^\mu r^\nu$ is positive definite on the future lightcone LC^+. In Appendix B, we prove that this is equivalent to the statement

that $B_{\mu\nu}$ is diagonalizable by an SO(1, 3) transformation, and after diagonalization it takes the form

$$B^{\mu\nu}=\begin{pmatrix} B_0 & 0 & 0 & 0 \\ 0 & -B_1 & 0 & 0 \\ 0 & 0 & -B_2 & 0 \\ 0 & 0 & 0 & -B_3 \end{pmatrix} \quad \text{with } B_0 > B_1, B_2, B_3. \tag{15}$$

We will refer to B_0 as the "timelike" eigenvalue of $B_{\mu\nu}$, and B_i, $i=1,2,3$, as its "spacelike" eigenvalues. The minus sign in front of the spacelike eigenvalues is the result of the Minkowski-space metric; see Appendix A. The degree of freedom in the definition of $B^{\mu\nu}$ amounts to shifting all the eigenvalues by the same constant and does not affect the inequalities (15). However, it can be used to manipulate the signs of the eigenvalues.

Finding the eigenvalues of $B^{\mu\nu}$ explicitly in terms of b_i requires a solution of a fourth-order characteristic equation, which constitutes one of the computational difficulties of straightforward algebra. We reiterate that in our analysis we never use these explicit expressions. Our analysis relies only on the fact that the eigenvalues are real and satisfy Eq. (15).

III. MINIMA OF THE LANDAU POTENTIAL

Having introduced the formalism that allows us to treat the most general 2OP GL model, let us proceed to the task of minimization of the free-energy functional. We do not consider here the effects of nontrivial boundary conditions, so we are looking for homogeneous solutions $\psi_i(\vec{r})=\langle\psi_i\rangle$ that minimize the Landau potential (6).

As mentioned in the Introduction, straightforward algebra is of little help for the minimization problem, since the resulting system of coupled equations $\partial V/\partial\psi_i=0$ cannot be solved in the general case. However, one can still learn much about the ground state of the general 2OP GL model without finding its location explicitly. In this paper, we will provide, in particular, answers to the following questions:

(i) How many extrema does the potential with given parameters have? How many of them are minima?

(ii) Can the global minimum be degenerate and when does it happen?

(iii) When does the global minimum spontaneously break an explicit symmetry of the potential?

(iv) What is the phase diagram of the model? What phase transitions can take place during continuous change of the coefficients of the model?

A. Number of extrema

Let us start with the number of extrema of a generic Landau potential. In order to find an extremum of V lying on the future lightcone LC^+, one can use the standard Lagrange multiplier method. In this case, one needs to introduce only one Lagrange multiplier λ, which leads to the following system:

$$B_{\mu\nu}\langle r^\nu\rangle - \lambda\langle r_\mu\rangle = A_\mu,$$

$$\langle r^\mu\rangle\langle r_\mu\rangle = 0. \tag{16}$$

Here, $\langle r^\mu\rangle$ labels the position of an extremum. To avoid cumbersome notation, we omit $\langle\cdots\rangle$ in this subsection.

To establish how many solutions system (16) has, consider the $B_{\mu\nu}$-diagonal frame (we remind the reader that for a potential stable in a strong sense, such a frame always exists). Then the first line in Eq. (16) takes the form

$$(B_0-\lambda)r_0=A_0, \quad (B_i-\lambda)r_i=A_i. \tag{17}$$

Rewriting $r_i=r_0n_i$, where n_i is a unit 3-vector, and eliminating λ, one obtains

$$[A_0-(B_0-B_i)r_0]n_i=A_i. \tag{18}$$

These three equations are coupled via the condition $|\vec{n}|=1$. Consider the left-hand side (l.h.s.) of Eq. (18) at fixed r_0 and all the unit vectors n_i. It parametrizes an ellipsoid with semiaxes

$$A_0-(B_0-B_1)r_0, \quad A_0-(B_0-B_2)r_0, \quad A_0-(B_0-B_3)r_0. \tag{19}$$

Now imagine how this ellipsoid changes if r_0 increases from zero to infinity. Let us for simplicity assume that the eigenvalues of $B^{\mu\nu}$ are distinct and $B_1<B_2<B_3$.

Assume first that $A_0>0$. Then, at $r_0=0$, Eq. (18) parametrizes a sphere with radius A_0. As r_0 increases, it turns into a continuously shrinking ellipsoid with semiaxes (19). At

$$r_0=r_0^{(1)}\equiv\frac{A_0}{B_0-B_1}$$

this ellipsoid collapses to the interior of a planar ellipse with semiaxes

$$A_0\frac{B_2-B_1}{B_0-B_1}, \quad A_0\frac{B_3-B_1}{B_0-B_1},$$

orthogonal to the first axis. As r_0 increases further, this ellipse returns to an ellipsoid with two shrinking semiaxes and one growing semi-axis, and at

$$r_0=r_0^{(2)}\equiv\frac{A_0}{B_0-B_2}$$

it collapses again to a flat ellipse with semiaxes

$$A_0\frac{|B_1-B_2|}{B_0-B_2}, \quad A_0\frac{B_3-B_2}{B_0-B_2},$$

orthogonal to the second axis. Further on, at $r_0=r_0^{(3)}$ it collapses to an ellipse orthogonal to the third axis, and for even larger values of r_0 this ellipsoid increases infinitely.

For each r_0 interval, the ellipsoid sweeps a certain region in the three-dimensional space.

(i) During the first stage, $0<r_0<r_0^{(1)}$, it sweeps the interior of the sphere of radius A_0, passing through each point exactly once.

(ii) During the second stage, $r_0^{(1)}<r_0<r_0^{(2)}$, it sweeps a certain region, bounded by the caustic surface shown in Fig. 1, left. It can be shown that each point inside this region is swept exactly *twice*.

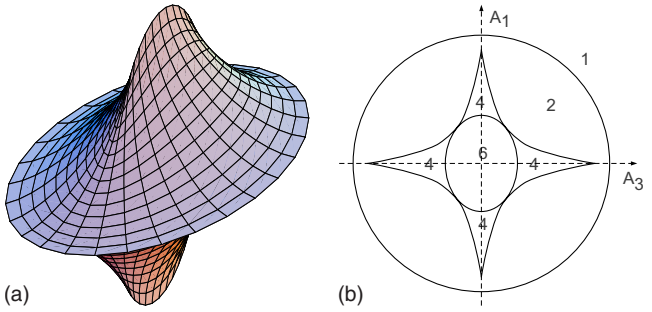

FIG. 1. (Color online) Left: The envelope of ellipsoids for $r_0^{(1)} < r_0 < r_0^{(2)}$. Right: (A_1, A_3) section of the caustic surfaces in the A_i space and of the sphere with radius A_0. The number of solutions of Eq. (18) is indicated for each region.

(iii) During the third stage, $r_0^{(2)} < r_0 < r_0^{(3)}$, it sweeps twice a similar caustic region, but oriented differently.

(iv) Finally, during the fourth stage, $r_0 > r_0^{(3)}$, it sweeps once the entire 3D space.

Note that the appearance of caustic regions in the potential extremization problem is natural, since this problem is known to exhibit some catastrophe theory phenomena; see, e.g., [4].

Returning to the system (18), which equates the l.h.s. to the 3-vector A_i, one sees that in order get the number of solutions of Eq. (18) without finding them explicitly, one simply has to check whether A_i falls inside these regions. Figure 1, right, illustrates this statement. It shows regions with different numbers of extrema on the (A_1, A_3) plane for $A_2 = 0$ and some $A_0 > 0$.

For the second possibility, $A_0 < 0$, the situation is much simpler. At $r_0 = 0$ we again start with the sphere of radius $|A_0|$. As r_0 grows, it turns into an ellipsoid with growing semiaxes (which is due to $B_0 - B_i > 0$), and it sweeps once the entire space outside the sphere.

The size of the above 3D regions is proportional to A_0. Therefore, in the (1+3)-dimensional space of 4-vectors A^μ, they define the corresponding conical regions starting from the origin. Therefore, the number of extrema of the potential depends on where the 4-vector A^μ lies:

(i) If A^μ lies inside the *past lightcone* LC^- (i.e., $A_0 < 0$ and $|\vec{A}| < |A_0|$), then system (16) has *no solution*. In this case, the quadratic term of the potential, $-A_\mu r^\mu$, increases in all directions in the ψ_i space. The only extremum of the potential is the global minimum at the origin, which corresponds to the high-symmetry ground state of the model.

(ii) If A^μ lies outside LC^-, then *at least one* nontrivial solution exists. If $A_0 < 0$ in the $B^{\mu\nu}$-diagonal basis, i.e., A^μ still lies in the lower hemispace, then this solution is unique and is the global minimum of the potential.

(iii) If A^μ lies inside LC^+, then *at least two* nontrivial extrema exist.

(iv) If A^μ lies inside one or both caustic cones defined above, then *two additional extrema per cone* appear.

In total, there can be up to six nontrivial extrema of the potential in the orbit space. This result was also found independently in [30] with a more traditional analysis of the Higgs potential of 2HDM. The largest number of extrema is realized in situations when $A_0 > 0$ and A_i is sufficiently small, so that A^μ lies inside both caustic cones.

Special care must be taken when r_0 of an extremum is exactly equal to one of the values $r_0^{(i)}$. Then the sequence of intersections of the ellipsoid with a given point A_i changes, but the overall counting rules given above remain the same. As we will see later, this situation corresponds to spontaneous violation of a discrete symmetry.

B. Number of local minima

In general, the above construction cannot distinguish a local minimum from a saddle point or a maximum, so other methods must be used to establish the number of local minima.

First of all, let us note that potential (6) with restrictions (15) cannot have nontrivial maxima [20,28]. This can be easily seen by drawing any ray in the ψ_i space from the origin and observing that the potential along this ray can be written as $\alpha|\psi|^2 + \beta|\psi|^4$ with $\beta > 0$. This function can never have a nontrivial maximum. Thus, the problem reduces to distinguishing minima from saddle points (in the orbit space).

Take a generic extremum of the potential in the ψ_1, ψ_2 space, and calculate the second derivative matrix of the potential (the Hessian) at this point,

$$(\Omega^2)_{\alpha\beta} = \frac{\partial^2 V}{\partial\phi_\alpha \partial\phi_\beta}. \tag{20}$$

Here, ϕ_α are the four real degrees of freedom, real and imaginary parts of ψ_1 and ψ_2,

$$\psi_1 = \phi_1 + i\phi_2, \quad \psi_2 = \phi_3 + i\phi_4. \tag{21}$$

We will refer to the eigenvalues of this matrix as "eigenfrequencies," ω_a^2. Due to the U(1) invariance of the potential, it always has one flat direction with zero eigenfrequency (one Goldstone mode), while among the other three, there is at least one positive eigenfrequency. Let us call the signs of these three eigenfrequencies (i.e., $+++$, $++-$, or $+--$) the *signature* of the Hessian.

Among the four degrees of freedom in the ψ_i space, three correspond to variations in the orbit space, i.e., to shifts of the point r^μ on LC^+ away from the extremum. If the extremum is not at the origin, then these shifts are linear functions of the shifts in the ψ_i space, and the Jacobian corresponding to this transformation is regular. Indeed, with the notation (21), one gets

$$\frac{1}{2}\frac{\partial r^\mu}{\partial\phi_\alpha} = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_3 & \phi_4 & \phi_1 & \phi_2 \\ \phi_4 & -\phi_3 & -\phi_2 & \phi_1 \\ \phi_1 & \phi_2 & -\phi_3 & -\phi_4 \end{pmatrix}. \tag{22}$$

If Φ_1 is non zero, then this matrix has one and only one zero eigenvalue, with the corresponding right eigenvector $(-\phi_2, \phi_1, -\phi_4, \phi_3)$ being the Goldstone mode. This can be seen most easily in the frame where $r^\mu \propto (1,0,0,1)$, implying $\phi_3 = \phi_4 = 0$ (obviously, such a frame always exists for any r^μ). The signature of the Hessian, therefore, is the same in the ψ_i space and in the orbit space.

For an extremum to be minimum, its signature must be $+++$. However, since the explicit expressions for $\langle r^\mu \rangle$ can-

not be given in the general case, checking this explicitly at each extremum is also problematic. One can circumvent this computational difficulty using the following Proposition.

Proposition 1. For each extremum, the Hessian remains signature-definite within each conical region described in the previous subsection.

Proof. Let us fix $B^{\mu\nu}$ and move A^μ continuously in the parameter space, keeping it strictly inside one of the regions described in the previous subsection. Let us pick up an extremum and follow how it changes when A^μ moves. Its position, its depth, as well as the eigenfrequencies are algebraic functions of the components of A^μ and therefore they also change continuously. So, if the Hessian changes signature along at the end points of some A^μ path, then there exists a point at which one of the eigenfrequencies is zero. Thus, the expansion of the potential at this point starts from the third- or fourth-order term, and this point corresponds to merging of two or three simple extrema, respectively.

But such a merging cannot happen for A_μ lying strictly inside the caustic regions. Indeed, if two sufficiently close points r_a^μ and r_b^μ are both extrema of the potential, then their respective zeroth components r_{a0} and r_{b0} are also close, so the intersection points of the corresponding ellipsoids lie close to the boundary of a caustic region. In the limit $r_a^\mu \to r_b^\mu$, the intersection points, A_μ being among them, approach the envelope (loosely speaking, the envelope can be viewed as the locus of intersections of the "successive" ellipsoids). ■

So, since the signature of the Hessian remains the same for all A_μ inside some region, one can select some representative A_μ, calculate the signature of the Hessian for it, and then extrapolate the results for all the points inside this region.

Let us now calculate the number of minima inside the innermost region of the A^μ space; see Fig. 1, right. For the representative point in this region, $A^\mu=(A_0,0,0,0)$, calculations can be easily done explicitly. Indeed, it follows from Eq. (18) that there are three pairs of extrema at $r_0=r_0^{(1)}$, $r_0^{(2)}$, and $r_0^{(3)}$,

$$r_0^{(1)}(1,\pm 1,0,0), \quad r_0^{(2)}(1,0,\pm 1,0), \quad r_0^{(3)}(1,0,0,\pm 1). \tag{23}$$

Again, let us order the eigenvalues B_i, $B_1<B_2<B_3$ and expand the potential near the point $\langle r^\mu\rangle_+=r_0^{(3)}(1,0,0,1)$. If $r^\mu=r_0(1,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)=\langle r^\mu\rangle_+ + \delta r^\mu$, then

$$\begin{aligned} V &= -A_0 r_0 + \frac{1}{2} r_0^2 (B_0 - B_1 \sin^2\theta\cos^2\phi \\ &\quad - B_2 \sin^2\theta\sin^2\phi - B_3\cos^2\theta) \\ &\approx -\frac{A_0^2}{2(B_0-B_3)} + \frac{B_0-B_3}{2}(\delta r_0)^2 \\ &\quad + \frac{A_0^2\theta^2}{2}\frac{(B_3-B_1)\cos^2\phi + (B_3-B_2)\sin^2\phi}{(B_0-B_3)^2}. \end{aligned} \tag{24}$$

Here, δr_0 and θ are small while ϕ can be arbitrary. Since B_3 is the largest spacelike eigenvalue, this point is a minimum, and so is the other extremum of this pair, $\langle r^\mu\rangle_-=r_0^{(3)}(1,0,0,-1)$. The same calculation for the extrema at $r_0=r_0^{(1)}$ and $r_0^{(2)}$ shows that they are saddle points. Thus, we find that for A_μ lying in the innermost region, the potential has two separate minima and four separate saddle points in the orbit space.

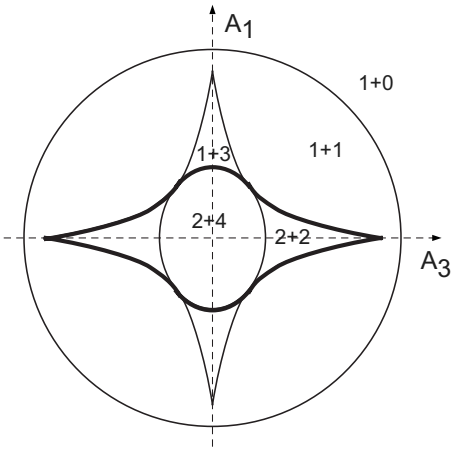

FIG. 2. The same as in Fig. 1, right, but with the number of minima and saddle points shown separately as $N_{\text{minima}}+N_{\text{saddle}}$. Thick lines show the section of the principal caustic cone.

As A_μ moves out of this region, the number of minima does not increase. Indeed, one can show that crossing the caustic surface at a generic point leads to the disappearance of two saddle points or of one saddle point and one minimum, but it cannot, for example, lead to the disappearance of three saddle points and the appearance of a new minimum. This can also be verified with the straightforward calculations similar to Eq. (24) by selecting points A_μ lying on the axes (for this choice, all the extrema can also be studied with explicit algebra). Therefore, we arrive at the following Proposition:

Proposition 2. The most general quadratic plus quartic potential with two order parameters can have at most two distinct local minima in the orbit space.

C. The principal caustic cone

In Sec. III A, we showed that for a generic potential there exist two caustic cones in the A^μ space. If $B_1<B_2<B_3$, they correspond to $r_0^{(1)}\leqslant r_0\leqslant r_0^{(2)}$ and $r_0^{(2)}\leqslant r_0\leqslant r_0^{(3)}$, respectively. The analysis of Sec. III B shows that these two caustic cones play different roles. It is the second cone, with $r_0^{(2)}\leqslant r_0\leqslant r_0^{(3)}$, which we call the *principal caustic cone*, that separates regions with a different number of *minima*. This is illustrated by Fig. 2, which is an updated version of Fig. 1, right, with the numbers of minima and saddle points shown separately. The most straightforward proof is based on the stability analysis of the extrema in three situations with A_μ lying on each of the three axes and Proposition 1. The other cone just separates regions with different numbers of saddle points and does not affect directly the properties of the global minimum.

D. Geometric reformulation of the search for the global minimum

Consider again the potential term in Eq. (13),

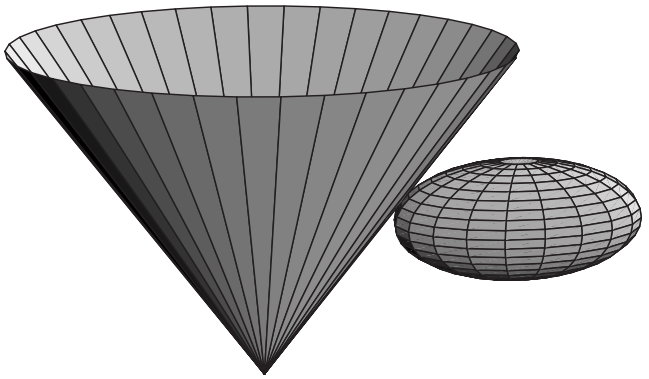

FIG. 3. A (1+2)-dimensional illustration of the contact between $\mathcal{M}^{C_{\min}}$ and LC$^+$. Shown is the case of a^μ lying outside LC$^+$.

$$V = -A_\mu r^\mu + \frac{1}{2}B_{\mu\nu}r^\mu r^\nu. \tag{25}$$

Let us exploit the freedom in definition of $B_{\mu\nu}$ to make $B_0 > 0$ and all $B_i < 0$. Then $(B^{-1})_{\mu\nu}$ exists, and Eq. (25) can be rewritten as

$$V = \frac{1}{2}B_{\mu\nu}(r^\mu - a^\mu)(r^\nu - a^\nu) + V_0, \quad a_\mu = (B^{-1})_{\mu\nu}A^\nu,$$

$$V_0 = -\frac{1}{2}(B^{-1})_{\mu\nu}A^\mu A^\nu. \tag{26}$$

Let us now define an *equipotential surface* $\mathcal{M}^C$ as a set of all vectors p^μ in the Minkowski space $\mathcal{M}$ such that

$$B_{\mu\nu}p^\mu p^\nu = B_0 p_0^2 + \sum_i |B_i| p_i^2 = C. \tag{27}$$

One sees that equipotential surfaces exist for $C \geqslant 0$ and are 3-ellipsoids nested into each other, with their eigenaxes aligned in the $B^{\mu\nu}$-diagonal frame along the eigenaxes of $B_{\mu\nu}$.

Returning to the potential (26), we see that C is related to the values of the potential: $C=2(V-V_0)$. Therefore, finding points in the orbit space with the same value of V amounts to finding intersections of the corresponding $\mathcal{M}^C$ with the forward lightcone LC$^+$. In particular, to find a local minimum of the potential in the orbit space, one has to find an equipotential surface that touches LC$^+$ (we say that two surfaces "touch" if they have parallel normals at the intersection points). The *global* minimum corresponds to the unique equipotential surface $\mathcal{M}^{C^{\min}}$ that only touches but never intersects LC$^+$.

Thus, the geometric strategy for the minimization of the potential is the following: (i) Construct a family of 3-ellipsoids $\mathcal{M}^C$ at the base point a^μ; (ii) find the unique 3-ellipsoid $\mathcal{M}^{C_{\min}}$ that merely touches LC$^+$ but never intersects it; (iii) the contact point or points give the values of r^μ (hence, ψ_1 and ψ_2) that minimize the potential.

To facilitate the visualization, Fig. 3 shows a (1+2)-dimensional analogue of the contact between $\mathcal{M}^{C_{\min}}$ and LC$^+$. In this particular example, a^μ, which is located at the center of the ellipsoid, lies outside LC$^+$.

Alternatively, using Eq. (27) one can interpret the potential as the distance squared from the point a^μ in the Euclidean metric diag $(B_0, |B_1|, |B_2|, |B_3|)$. The minimization problem is then reformulated as a search for points on LC$^+$ that are closest to a^μ in this metric. Since the forward lightcone LC$^+$ (together with its interior) is a concave region, this representation immediate leads to the following conclusion: the necessary condition for the existence of a degenerate minimum is that a^μ lies inside LC$^+$: $a^\mu a_\mu > 0$. Later, in Sec. V C we will give necessary and sufficient conditions for this to happen.

The two geometric constructions described above, the ones based on the equipotential surfaces and on the caustic cones, are related to each other in the same manner as a planar curve to its evolute. To illustrate this relation, let us consider a simple planar problem: find on the unit circle points of local minima of the "potential" $V=B_{ij}(n_i-a_i)(n_j-a_j)$ with $B_{ij}=\mathrm{diag}(B_1,B_2)$, $B_1 \neq B_2$, $B_i > 0$. In the coordinates $\tilde{n}_i=(\sqrt{B})_{ij}n_j$, the "potential" becomes $V=|\tilde{n}_i-\tilde{a}_i|^2$, while the unit circle is transformed into an ellipse. One can easily verify that the number of local minima depends on whether point $\tilde{a}^i$ lies inside the evolute of this ellipse.

IV. POTENTIALS STABLE IN A WEAK SENSE

Let us now discuss how the above constructions change for a potential stable in a weak sense, i.e., a potential whose V_4 can have flat directions in the ψ_i space, along which the potential is stabilized by the V_2 term.

A flat direction of the quartic part of the potential in the ψ_i space corresponds to a vector $r^\mu \in$ LC$^+$ in the orbit space such that $B_{\mu\nu}r^\mu r^\nu = 0$. Such an r^μ must be an eigenvector of $B^{\mu\nu}$; see Appendix B. Let us first assume that there is only one such direction. Aligning it with the first axis, one can diagonalize $B^{\mu\nu}$ in the "transverse space," bringing it to the following generic form:

$$B^{\mu\nu} = \begin{pmatrix} B_0+\delta B & \delta B & 0 & 0 \\ \delta B & -B_0+\delta B & 0 & 0 \\ 0 & 0 & -B_2 & 0 \\ 0 & 0 & 0 & -B_3 \end{pmatrix}$$
$$\text{with } \delta B \geqslant 0, B_0 > B_2, B_3. \tag{28}$$

Note that in contrast to the potentials stable in a strong sense, Eq. (15), this $B^{\mu\nu}$ cannot be diagonalized by an SO(1, 3) transformation. Indeed, a boost along the first axis with "rapidity" η leads to the same $B^{\mu\nu}$ as Eq. (28) but with redefined $\delta B \to e^{-2\eta}\delta B$ (see Appendix A). If $\delta B \neq 0$, then $B^{\mu\nu}$ is not diagonalizable.

To find the number of extrema in this case, one can again start with system (16), but instead of considering fixed A_0 sections in the A_μ space one can fix one of its lightcone components. Let us introduce the lightcone decomposition of any 4-vector,

$$p^\mu = p_+ n_+^\mu + p_- n_-^\mu + p_\perp^\mu, \quad n_\pm^\mu = (1, \pm 1, 0, 0), \tag{29}$$

where $p_\perp^\mu = (0, 0, p_2, p_3)$. The lightcone coordinates $p_\pm$ are related to the zeroth and first coordinates p_0, p_1 as $p_\pm$

$=(p_0 \pm p_1)/2$. Then, system (16) can be rewritten as

$$(B_0 - \lambda) r_- = A_-,$$

$$(B_0 - \lambda) r_+ + 2\delta B r_- = A_+,$$

$$(B_i - \lambda) r_i = A_i, \quad i = 2,3,$$

$$4 r_+ r_- = r_2^2 + r_3^2. \tag{30}$$

The condition that the quadratic part of the potential $V_2 = -A_\mu r^\mu$ increases along the flat direction of V_4 given by n_+^μ implies that $A_- < 0$. Repeating the geometric analysis described in full detail for the potential stable in a strong sense, we arrive at the following conclusion:

(i) If A^μ lies inside the past lightcone LC^-, then system (30) has no solution. The global minimum is at the origin.

(ii) If A^μ lies outside LC^- (but still with $A_- < 0$), then system (30) has a unique solution. The corresponding unique extremum of the potential is the global minimum.

Suppose now that there is more than one flat direction of V_4. Let us pick up two such distinct vectors r_1^μ, $r_2^\mu \in \mathrm{LC}^+$, both eigenvectors of $B_{\mu\nu}$,

$$B_{\mu\nu} r_1^\nu = \lambda_1 r_{1\mu}, \quad B_{\mu\nu} r_2^\nu = \lambda_2 r_{2\mu}.$$

Since $r_1^\mu r_{2\mu} \neq 0$, one obtains $\lambda_1 = \lambda_2$. Then, using the freedom in definition of $B_{\mu\nu}$, one can always set this common eigenvalue to zero. Then for any linear combination of these two vectors, one gets

$$B_{\mu\nu}(\alpha r_1 + \beta r_2)^\nu = 0. \tag{31}$$

Consider now such an SO(1, 3) transformation that makes $r_1^\mu \propto n_+^\mu$ and $r_2^\mu \propto n_-^\mu$. Then, $B_{\mu\nu}$ that satisfies Eq. (31) takes the following generic form:

$$B^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -B_2 & 0 \\ 0 & 0 & 0 & -B_3 \end{pmatrix} \quad \text{with } 0 \geqslant B_2, B_3. \tag{32}$$

This form is diagonal [note also that it is equivalent to Eq. (28) with $\delta B = 0$], so one can again switch to the fixed A_0 sections. Since $-A_\mu r^\mu$ must increase along $n_\pm^\mu$, $A_0 < 0$. Repeating the same analysis as in Sec. III A, one finds that the potential has a unique nontrivial extremum if $|A_1| < |A_0|$ and $A^\mu A_\mu < 0$.

So, a potential stable in a weak sense is similar to the potentials stable in a strong sense with $A_0 < 0$ in the frame with diagonal $B_{\mu\nu}$. It can have no more than one nontrivial extremum, which is then necessarily the global minimum.

V. SYMMETRIES AND THEIR VIOLATION

A. Explicit symmetries

As explained in Sec. II A, the free energy remains invariant under an appropriate simultaneous transformation of the order parameters ψ_i *and* the coefficients. It can happen, however, that the free energy is invariant under some specific transformation of ψ_i (or the coefficients) *alone*. We call this symmetry an *explicit symmetry* of the free energy.

In the orbit space, this symmetry corresponds to such a map of the Minkowski space $\mathcal{M}$ that leaves invariant, separately, $B_{\mu\nu} r^\mu r^\nu$, $A_\mu r^\mu$, and $K_\mu \rho^\mu$. The notion of explicit symmetry is invariant under the Lorentz group of the orbit space transformations; so, any SO(1, 3) transformation leaves a given model in the same symmetry class.

In simple cases, the presence of a symmetry can be evident from a direct inspection of the free-energy functional, see, e.g., examples in Sec. VII. In fact, in many concrete applications, the Landau potential is even constructed in such a way that some symmetry is preserved. In the general case, however, a nonevident hidden symmetry can exist even in complicated forms of the free energy, without being easily noticeable. So, one needs a reparametrization-invariant criterion that can help recognize the presence of a symmetry using only K_μ, A_μ, and $B_{\mu\nu}$. In addition, it would also useful to know what this symmetry is.

Both questions are answered by the following Proposition:

Proposition 3. Suppose that the free energy (4) is explicitly invariant under some transformations of r^μ. Let G be the maximal group of such transformations. Then (a) G is nontrivial if and only if there exists an eigenvector of $B_{\mu\nu}$ orthogonal both to A_μ and K_μ; (b) group G is one of the following groups: Z_2, $(Z_2)^2$, $(Z_2)^3$, O(2), $\mathrm{O}(2) \times Z_2$, or O(3), and depends on the number of the eigenvectors of $B_{\mu\nu}$ to which A_μ and K_μ are orthogonal, and on whether $B_{\mu\nu}$ has degenerate eigenvalues.

Proof. Let us start with the potential stable in a strong sense. Let us denote the group of all explicit symmetries of $B_{\mu\nu}$, A_μ, and K_μ by G_B, G_A, and G_K, respectively. Obviously,

$$G = G_B \cap G_A \cap G_K. \tag{33}$$

The group of explicit symmetries is necessarily a subgroup of the O(3) transformation group of the three-dimensional space in the $B_{\mu\nu}$-diagonal frame; so one can switch to the spacelike parts only (B_{ij}, A_i, K_i).

Consider now G_B. If all spacelike eigenvalues of $B_{\mu\nu}$ are different, then its only symmetries are reflections of each of the spacelike eigenaxes, which generate group $G_B = (Z_2)^3$. If two eigenvalues coincide, then $G_B = \mathrm{O}(2) \times Z_2$, and if all three of them are equal, then $G_B = \mathrm{O}(3)$. Note that in all of these cases the following statement holds: if some Z_2 group is a subgroup of G_B, then its generator flips the direction of some eigenvector of B_{ij}.

Similarly, G_A is O(2) (rotations around the axis defined by A_i) if A_i is a nonzero vector, and O(3) otherwise. The same holds for K_i, the only difference being the direction of the axis. If we want G to be nontrivial, then the lowest possible symmetry of A_i and K_i together (given by a Z_2 group) must also be a symmetry of B_{ij}, i.e., it must flip one of the eigenvectors of B_{ij}. In other words, both A_i and K_i are orthogonal to this eigenvector. Since this purely spacelike eigenvector is also the eigenvector of $B_{\mu\nu}$, we arrive at the first statement of this Proposition.

Detailed classification depends on the *number* of eigenvectors of B_{ij} that are orthogonal to A_i and K_i.

(i) If A_i and K_i are orthogonal to all three eigenvectors ($A_i=K_i=0$), then $G=G_B$.

(ii) If A_i and K_i are orthogonal to two eigenvectors ($A_i \| K_i$ and are themselves eigenvectors of B_{ij}), then $G=(Z_2)^2$ or O(2).

(iii) Finally, if there is only one eigenvector of B_{ij} orthogonal both to A_i and K_i, then the symmetry group is Z_2.

For a potential stable in a weak sense, we first note that the eigenvectors of $B^{\mu\nu}$ are either the lightcone vectors or purely spacelike eigenvectors. Since K^μ lies inside LC^+, it cannot be orthogonal to any lightcone vector. Therefore, one has to check the above conditions only for the "transverse" eigenvectors, which reduces the above list of possible symmetry groups to Z_2, $(Z_2)^2$, O(2). ■

The necessary and sufficient condition formulated in the first part of this Proposition can be written in a reparametrization-invariant way. The method is essentially the same as in [31,32] and is based on a simple observation: if a 3-vector a_i is orthogonal to some eigenvector of a real symmetric matrix b_{ij}, then the triple scalar product of vectors a_i, $b_{ij}a_j$, and $b_{ij}b_{jk}a_k$ is zero. In Minkowski space, we introduce

$$K_{0\mu} \equiv K_\mu, \quad K_{1\mu} \equiv B_\mu{}^\nu K_\nu, \quad K_{2\mu} \equiv (B^2)_\mu{}^\nu K_\nu,$$

$$K_{3\mu} \equiv (B^3)_\mu{}^\nu K_\nu, \tag{34}$$

where B^k is the kth power of $B_{\mu\nu}$. The same series can be written for A_μ. For any four 4-vectors a^μ, b^μ, c^μ, and d^μ, we introduce the short-hand notation

$$(a,b,c,d) \equiv \epsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma.$$

Then the condition "there exists an eigenvector of $B_{\mu\nu}$ orthogonal to K_μ" can be written as

$$(K_0,K_1,K_2,K_3) = 0. \tag{35}$$

Note that since K^μ always lies inside the future lightcone, it can be orthogonal only to spacelike eigenvectors of $B_{\mu\nu}$, which is exactly what is needed. Then, the statement of Proposition 3a can be reproduced if we accompany Eq. (35) with a similar condition for A_μ,

$$(A_0,A_1,A_2,A_3) = 0, \tag{36}$$

and with the condition that these two 4-vectors be orthogonal to the *same* eigenvector of $B_{\mu\nu}$, for example,

$$(A_0,A_1,A_2,K_0) = 0. \tag{37}$$

Conditions (35)–(37) can be straightforwardly checked in any basis once $B_{\mu\nu}$, A_μ, and K_μ are known. Thus, the presence of any hidden symmetry can be verified without the need to find this symmetry explicitly.

B. Symmetries of the potential versus symmetries of the free energy

Explicit symmetries of the entire free energy depend on B_{ij}, A_i, and K_i, while the symmetries of the potential depend only on B_{ij} and A_i. Therefore, it might happen that the potential has a larger symmetry group than the entire free energy. A simple example is

$$F = \kappa(|\vec{D}\psi_1|^2 + |\vec{D}\psi_2|^2) + 16\lambda(|\psi_1|^2 - v^2)^2 + \lambda(|\psi_2|^2 - 4v^2)^2. \tag{38}$$

The potential here is symmetric under $\psi_2 \leftrightarrow 2\psi_1$, while the gradient term is not.

The two notions, i.e., the symmetry of the potential or of the entire free energy, play different roles. When one seeks for the minimum of the Landau potential, the coefficients in the gradient term (K^μ) are irrelevant. However, the symmetry of the spectrum of small oscillations of the order parameters above the ground state is the one of the entire free-energy functional.

C. Spontaneous breaking of an explicit symmetry

Even if the Landau potential is invariant under some transformation of Φ, the values $\langle\Phi\rangle$ that minimize it do not necessarily have to preserve the same symmetry. In the orbit space, if the Landau potential is invariant under a group G of transformation of r^μ, then the position of the global minimum might be invariant only under a proper subgroup of G. In such situations, one speaks of spontaneous breaking of the symmetry. Since the *set* of all global minima is invariant under the full explicit symmetry group G, the spontaneous breaking of an explicit symmetry always leads to degenerate global minima.

For our problem, several results follow immediately from Proposition 2:

(i) The global minimum can be only twice degenerate.

(ii) Minima that preserve and violate any discrete symmetry cannot coexist.

(iii) The maximal breaking of any discrete symmetry consists in removing only one Z_2 factor: $(Z_2)^k \to (Z_2)^{k-1}$ with $k=1,2,3$.

In addition, in [29] it was proven that the twice degenerate global minimum of Landau potential with quadratic and quartic terms is *always* realized via spontaneous breaking of some explicit Z_2 symmetry of the potential (but not necessarily of the entire free energy).

Let us now consider the question of *when* a given explicit symmetry is broken, focusing on the discrete symmetry case.

First of all, the global minimum must be degenerate. This immediately leads to the conclusion that the spontaneous violation can take place only in potentials stable in a strong sense, and in addition, only when A_μ lies inside the principal caustic cone. To make the discussion concrete, consider A_μ and K_μ in the $B_{\mu\nu}$-diagonal frame. Suppose that all B_i are distinct and the components $A_3=K_3=0$, while the other components are nonzero. Then, the free energy has an explicit Z_2 symmetry generated by reflections of the third coordinate. This explicit symmetry is conserved if the global minimum is at $r^\mu=(r_0,r_1,r_2,0)$, and it is spontaneously broken if the two degenerate global minima are at $r^\mu_\pm=(r_0,r_1,r_2,\pm r_3)$ with $r_3 \neq 0$.

Let us now recall the "shrinking ellipsoid" construction of Sec. III A. A degenerate extremum implies that two distinct points $n_{i\pm}$, when inserted in system (18), give the same point $A_i=(A_1,A_2,0)$ at the same r_0. This happens only when $r_0=r_0^{(3)}$ and the planar vector (A_1,A_2) lies inside the ellipse with semiaxes

$$A_0\frac{|B_1-B_3|}{B_0-B_3},\quad A_0\frac{|B_2-B_3|}{B_0-B_3}.$$

Besides, as we prove in Appendix C, in order for this extremum to be minimum, B_3 must be the largest (i.e., the closest to B_0) eigenvalue among all B_i. Thus, one arrives at the following necessary and sufficient reparametrization-invariant criterion for the spontaneous violation of a Z_2 symmetry (along the third axis):

$$B_3 > B_1, B_2 \text{ and } \frac{A_1^2}{(B_3-B_1)^2}+\frac{A_2^2}{(B_3-B_2)^2}<\frac{A_0^2}{(B_0-B_3)^2}. \tag{39}$$

It immediately follows from here that a^μ defined in Sec. III D lies inside the forward lightcone.

Finally, we would like to stress one important point. What is relevant for the whole discussion is the group G of explicit symmetries and its reduction upon symmetry breaking, but not the particular *realization* of the transformations of this group. For example, the free energy can be symmetric under $\psi_i \to \psi_i^*$ transformation or under $\psi_1 \leftrightarrow \psi_2^*$ transformation. These seemingly distinct Z_2 symmetries are, in fact, just different realizations of the *same symmetry class*. This becomes evident in the orbit space, as the former transformation corresponds to the flip of the second axis, while the latter corresponds to the flip of the third axis; so, both models are related by a reparametrization transformation. Therefore, all properties of spontaneous violation of these two particular sorts of the Z_2 symmetry are in fact identical.

Thus, there exists a reparametrization-invariant class of Z_2-symmetric models, a reparametrization-invariant class of $(Z_2)^2$-symmetric models, etc. Our analysis applies to *all* particular realizations of a given symmetry in contrast. This rather simple fact illustrates the usefulness of considering the most general GL model and might lead to establishing direct links between seemingly unrelated models.

VI. PHASE DIAGRAM AND PHASE TRANSITIONS

A. Phase diagram

The results obtained in the previous sections allow us to describe explicitly the phase diagram of the most general model with two order parameters in the mean-field approximation. We do this by classifying the phases according to the symmetries of the free-energy functional and to the properties of the ground state. For definiteness, we again sort the eigenvalues B_i as $B_1 \leqslant B_2 \leqslant B_3$. (i) Potential is stabilized by the quadratic term (A^μ lies inside LC$^-$). The global minimum is at the origin, $\psi_1=\psi_2=0$; this is the high-symmetry phase. (ii) Potential stable in a weak sense. The global minimum is always nondegenerate and preserves any explicit symmetry of the free energy. (iii) Potential stable in a strong sense. $B^{\mu\nu}$ can be diagonalized, and one can work with its spacelike part.

(1) All B_i are distinct.

(a) A_i and K_i are generic vectors (not orthogonal to any eigenvector of B_{ij}). No explicit symmetry is present. There can be one or two nondegenerate minima, depending on whether A_i lies inside the principal caustic region.

(b) A_i and K_i are both orthogonal to one eigenvector of B_{ij}. The explicit symmetry group is Z_2. The ground state can either break or preserve this symmetry. The symmetry is broken if it is the third axis that A_i and K_i are orthogonal to (i.e., $A_3=K_3=0$) and if condition (39) is satisfied.

(c) A_i and K_i are both parallel to the same eigenvector of B_{ij}. The explicit symmetry group is $(Z_2)^2$. The ground state can either preserve this symmetry or break it to Z_2. The criterion of the symmetry breaking is the same, Eq. (39), apart from the fact that now one of A_1, A_2 is zero.

(d) A_i and K_i are orthogonal to all three eigenvectors of B_{ij} (i.e., $A_i=K_i=0$, $A_0>0$). The explicit symmetry group is $(Z_2)^3$. The global minimum is always twice degenerate and breaks this symmetry to $(Z_2)^2$.

(2) Two eigenvalues among B_i coincide. Case $B_1<B_2=B_3$. The principal caustic cone reduces to a segment along the first axis.

(a) A_i is not aligned along the first axis. Then, the global minimum is nondegenerate. If K_i lies in the $(e_{(1)i}, A_i)$ plane, then there is an explicit Z_2 symmetry that is preserved at the minimum.

(b) A_i is the eigenvector of B_{ij} along the first axis ($A_2=A_3=0$). Then, if condition (39) is satisfied, there is a continuum (namely, a circle) of degenerate minima, otherwise the global minimum is nondegenerate. If, in addition, $K_2=K_3=0$, then the explicit symmetry group is O(2), and in the case of symmetry breaking, it is broken to Z_2. Instead, if K_i is a generic vector, then the explicit symmetry group is Z_2, and it may be preserved or broken depending on which minimum in the continuum is selected.

(c) $A_i=K_i=0$, $A_0>0$. The explicit symmetry group is $O(2)\times Z_2$. There is always a continuum (a circle) of global minima, and the symmetry is broken to $(Z_2)^2$.

(3) Two eigenvalues among B_i coincide. Case $B_1=B_2<B_3$. The analysis is similar to the case with all distinct B_i, with the following differences:

(a) If A_i and K_i are both parallel to the eigenvector $e_{(3)i}$, the explicit symmetry group is O(2) and it is always preserved at the global minimum.

(b) If $A_i=K_i=0$, then the explicit symmetry group is $O(2)\times Z_2$, which is broken at the global minimum to O(2).

(4) All three eigenvalues B_i coincide.

For a generic pair of A_i and K_i, there is always an explicit Z_2 symmetry. It is promoted to the O(2) symmetry if A_i and K_i are parallel, and to the O(3) symmetry if $A_i=K_i=0$. The symmetry is always preserved, apart form the case $A_0=0$, when it is broken to O(2) or Z_2.

B. First- and second-order phase transitions

A remarkable property of the two-order-parameter model is that it can have a first-order phase transition even at zero temperature and in the mean-field approximation. This is entirely due to the coexistence of two local minima (in the orbit space) with different depths. If upon continuous change of the coefficients the relative depth of the two distinct minima changes sign (the shallower minimum becoming the deeper one), the system occupying initially the global minimum be-

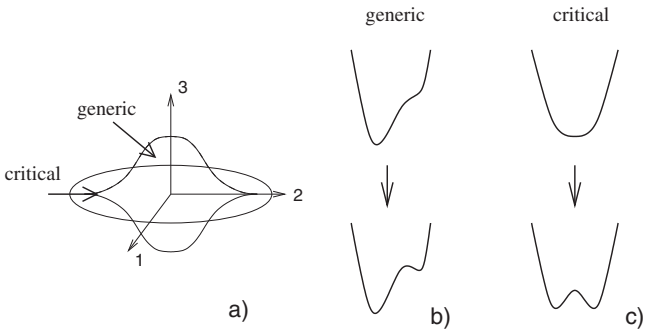

FIG. 4. (a) Schematic view of the principal caustic region. The two arrows enter it via a generic or a critical point. (b) Schematic change of the potential upon a generic entrance into the principal caustic cone. (c) Schematic change upon the entrance via a critical point.

comes metastable and can jump into the new global minimum via fluctuations or quantum tunneling.

If B_3 is nondegenerate, then the surface of first-order phase transitions in the A_i space is the interior of the ellipse at $r_0^{(3)}$, i.e., it is given by $A_i=(A_1,A_2,0)$, where A_1,A_2 satisfy Eq. (39). The border of this ellipse,

$$A_3=0, \quad \frac{A_1^2}{(B_3-B_1)^2}+\frac{A_2^2}{(B_3-B_2)^2}=\frac{A_0^2}{(B_0-B_3)^2},$$

is the critical line, at which the second-order phase transition takes place. If B_3 is degenerate, $B_1<B_2=B_3$, then the second-order phase transition takes place at isolated points,

$$A_3=A_2=0, \quad A_1=\pm A_0\frac{B_3-B_1}{B_0-B_3},$$

and the points of the first-order transitions form a linear segment between them. Finally, if all B_i are degenerate, then there is a single critical point at the origin, $A_i=0$, and there is no first-order phase transition.

Reconstruction of critical surfaces or lines in the A_μ space is obvious.

C. Critical properties: An example

It appears plausible that all the mean-field critical exponents of the general 2OP GL model are of a geometric nature and can be calculated without the knowledge of the exact position of the global minimum. Here, we do not explore this issue in full detail, but just illustrate it with one example: calculation of the critical exponent of the correlation length with the aid of differential geometry.

Let us fix $B^{\mu\nu}$ and move A^μ continuously in its parameter space. When it crosses the principal caustic cone (see Sec. III C), the number of minima changes. Two scenarios are possible; see Fig. 4(a). If A^μ enters the principal caustic cone through a generic point, then an additional local minimum appears together with an additional saddle point, as is shown schematically in Fig. 4(b). This bifurcation does not involve the global minimum. However, if A^μ enters the principal caustic cone through any of the critical points, then it is the global minimum that bifurcates into a minimum-saddle-minimum sequence, see Fig. 4(c).

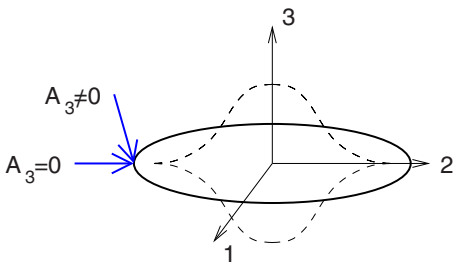

FIG. 5. (Color online) The principal caustic region in the A_i space (dashed line) and its border, the critical line (think solid line). The critical exponent δ depends on whether A_i approaches the critical line always staying in the plane $A_3=0$ or from outside the plane.

When A^μ approaches the critical surface or line, the corresponding eigenfrequency decreases and turns zero exactly at the critical surface. If the distance from A_i to the critical surface is $\varepsilon\to 0$, the eigenfrequency associated with this bifurcation decreases as

$$\omega^2 \propto \varepsilon^\delta. \tag{40}$$

Here we used the fact that the Jacobian of the map of non-Goldstone modes in ψ_i space to the surface of LC^+ is regular, if the extremum is not at the origin; see Eq. (22). The correlation length then behaves as $r_c \propto \varepsilon^{-\delta/2}$. We argue that the value of δ is of a geometric nature and can be calculated without the knowledge of the exact position of the minimum.

Let us first note that in the case of a single order parameter, $\delta=1$ simply because the eigenfrequency ω^2 is linearly proportional to the coefficient a in the potential (1). In the two-order-parameter case, due to the higher dimensionality of the A^μ space, one can approach a critical point from different directions.

As we described in Sec. III D, the search for the global minimum can be reformulated as a search for points lying on LC^+ that are closest to a given point a^μ in the Euclidean metric $\mathrm{diag}(B_0,|B_1|,|B_2|,|B_3|)$. In Appendix D, using a planar example, we show how to apply differential geometry to analyze the properties of the potential near a critical point. We showed, in particular, that the value of δ depends on the direction of approach to the critical point,

$$\text{generic direction} \to \delta=2/3,$$

$$\text{symmetric approach} \to \delta=1. \tag{41}$$

These exponents are robust in the sense that they remain the same for almost all (in the measure-theoretic definition) regular planar curves. It applies also to the second-order curves, which share the key property of the generic curves, namely that they have points of no more than fourth-order contact with a circle.

This technique can be extended to higher dimensions leading to the same results. So, exponents (41) apply to our problem as well, where the "symmetric approach" is understood as "A_i lying in the $A_3=0$ plane"; see Fig. 5.

It would be interesting to check the critical properties of all possible phase transitions in the 2OP GL model and see how many classes of critical behavior it can incorporate.

VII. EXAMPLES

Here we illustrate the general approach with several simple examples. Some of them are relevant for condensed-matter problems discussed in the literature.

A. Real coefficients

Consider a free-energy functional (4) with all real coefficients. It implies that the free energy remains invariant under simultaneous transformation $\psi_i \to \psi_i^*$, which corresponds in the orbit space to the reflection of the second coordinate,

$$r^\mu = (r_0, r_1, r_2, r_3) \to (r_0, r_1, -r_2, r_3). \tag{42}$$

Consequently, the 4-vectors and 4-tensor of the coefficients are

$$B^{\mu\nu} = \begin{pmatrix} \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot \\ 0 & 0 & -B_2 & 0 \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}, \quad A^\mu, K^\mu = (\cdot, \cdot, 0, \cdot), \tag{43}$$

where dots indicate generic values. Evidently, conditions (35)–(37) are satisfied.

Just to give a particular example, consider the free-energy functional of a two-gap superconductor in the dirty limit; see Eq. (49) in [14]. Its potential can be rewritten in the reparametrization-invariant way (13) with

$$B^{\mu\nu} = \frac{1}{2}\begin{pmatrix} \frac{b_1+b_2}{2} - b_i & -2b_i & 0 & -\frac{b_1-b_2}{2} \\ -2b_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{b_1-b_2}{2} & 0 & 0 & \frac{b_1+b_2}{2} - b_i \end{pmatrix},$$

$$A^\mu = \frac{1}{2}(-a_1 - a_2, a_i, 0, a_1 - a_2). \tag{44}$$

Here we used the notation of [14]: coefficients a_1, b_1 and a_2, b_2 refer to the properties of the first- and second-order parameters, respectively, while a_i, b_i describe interaction terms. The gradient terms considered in [14] are anisotropic, but they also contain real coefficients. As a result, A^μ and K^μ are orthogonal to the second eigenvector of $B^{\mu\nu}$. Note that in this example, the eigenvalue $B_2 = 0$. In order to find the other eigenvalues, one has to solve the cubic characteristic equation.

If the position of the global minimum has $\langle r_2 \rangle = 0$, then the symmetry is preserved, and there is no relative phase between the two order parameters. If $\langle r_2 \rangle \neq 0$, then the symmetry is spontaneously broken, and $\langle \psi_1 \rangle$ and $\langle \psi_2 \rangle$ in the ground state have a nonzero relative phase. In order to find whether spontaneous violation takes place, one has to diagonalize $B^{\mu\nu}$, find its eigenvalues as well as find A^μ in this frame, and then check inequality (39).

If one requires, in addition, that the order parameters ψ_i themselves be real, then $r_2 \equiv 0$, and the second axis can be omitted altogether. The orbit space is then simplified to the forward lightcone in the (1+2)-dimensional Minkowski space. Repeating the analysis of Sec. III, one obtains now at most four nontrivial extrema, among which up to two can be minima, and only one caustic cone.

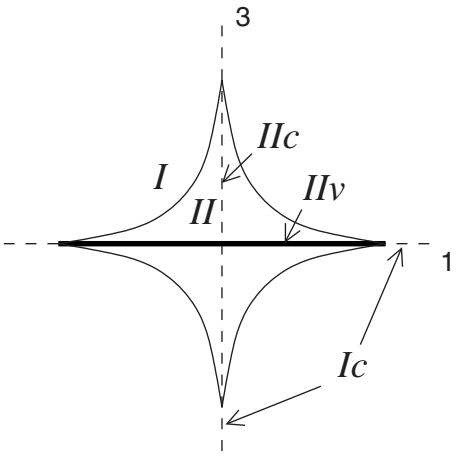

FIG. 6. Phases of the general Ginzburg-Landau model with two real order parameters on the A_i plane, classified according to the number of minima and conservation or violation of explicit symmetries. Shown is the case $B_1 < B_3$.

The phase diagram in this case is simpler. For example, in Fig. 6 we show phases in the A_i space for the case $B_3 > B_1$ and, for simplicity, we assume that $K_i = 0$. The astroid shown here is the planar analogue of the cusped region from Fig. 1 and Fig. 2. It separates the A_i regions corresponding to potentials with one minimum (phase I) and two minima (phase II). In addition, if A_i lies on the axes, the potential has an explicit symmetry. Dashed lines correspond to the cases in which the ground state conserves the symmetry (phases Ic and IIc), while the thick solid line corresponds to the phase that spontaneously breaks the discrete symmetry (phase IIv).

B. No ψ_1/ψ_2 mixing in the quartic potential: $b_5 = b_6 = 0$

The situation simplifies considerably if the quartic potential does not mix ψ_1 and ψ_2, i.e., when $b_5 = b_6 = 0$. In this case, $B^{\mu\nu}$ breaks into two 2×2 blocks and can be easily diagonalized by a boost along the third axis (see Appendix A for details) with "rapidity,"

$$\eta = \frac{1}{4} \log\left(\frac{b_1}{b_2}\right), \tag{45}$$

where we assumed $b_1 > b_2$, and by the rotation between the first and second axes by an angle equal to half of the phase of b_4. The resulting eigenvalues are

$$B_0 = \frac{1}{2}(\sqrt{b_1 b_2} + b_3), \quad B_{1,2} = \pm \frac{|b_4|}{2}, \quad B_3 = \frac{1}{2}(-\sqrt{b_1 b_2} + b_3). \tag{46}$$

The condition for the stability in a strong sense of the potential is

$$b_1 > 0, \quad b_2 > 0, \quad \sqrt{b_1 b_2} + b_3 > |b_4|. \tag{47}$$

If A^μ and K^μ are generic vectors, then the further analysis proceeds as in the generic case.

C. Interaction only via the $|\psi_1|^2|\psi_2|^2$ term

Let us assume now that the only interaction between the two OPs is given by the $b_3|\psi_1|^2|\psi_2|^2$ term. The 4-vector A^μ can be written as $(A_0,0,0,A_3)$, so that the potential has an explicit O(2) symmetry. The eigenvalues of $B^{\mu\nu}$ expressed in terms of the original coefficients are

$$B_0 = \frac{1}{2}(\sqrt{b_1 b_2} + b_3), \quad B_{1,2} = 0, \quad B_3 = \frac{1}{2}(-\sqrt{b_1 b_2} + b_3), \tag{48}$$

where for stability we require $\sqrt{b_1 b_2} + b_3 > 0$. In the frame where $B^{\mu\nu}$ is diagonal, A^μ takes the form

$$(\tilde{A}_0,0,0,\tilde{A}_3) = (A_0 \cosh\,\eta + A_3 \sinh\,\eta, 0, 0, A_3 \cosh\,\eta + A_0 \sinh\,\eta), \tag{49}$$

where η is given by Eq. (45).

If $B_3 > 0$, i.e., $b_3 > \sqrt{b_1 b_2}$, then the explicit O(2) symmetry is always conserved, since B_1 and B_2 are not the largest spacelike eigenvalues. The global minimum is at $\langle r^\mu \rangle \propto n_+^\mu$ or n_-^μ, which corresponds to

$$\langle \psi_1 \rangle \neq 0, \quad \langle \psi_2 \rangle = 0 \quad \text{or} \quad \langle \psi_1 \rangle = 0, \quad \langle \psi_2 \rangle \neq 0.$$

If $B_3 < 0$ and if A^μ lies inside the caustic cone, then this symmetry is spontaneously broken, and there exists a continuum of global minima with both $\langle \psi_1 \rangle \neq 0$ and $\langle \psi_2 \rangle \neq 0$ and an arbitrary relative phase between them. The condition that A^μ lies inside the caustic cone is

$$\left| \frac{\tilde{A}_3}{B_1 - B_3} \right| < \frac{\tilde{A}_0}{B_0 - B_1},$$

which, in terms of the original coefficients, translates into

$$a_1 \frac{b_3}{b_1} < a_2 < a_1 \frac{b_2}{b_3}. \tag{50}$$

Of course, the same bounds can be obtained by direct calculations.

Note that in the case of no interaction at all, $b_3 = 0$, the condition for the symmetry violation reads $|A_3| < A_0$. This means $a_1 > 0$, $a_2 > 0$, which is indeed expected.

VIII. SOLITONS

Two local order parameters can lead to the existence of solitons, i.e., states with nontrivial coordinate dependence of the mean-field values of the order parameters $\langle \psi_i \rangle(\vec{r})$ stable against small variations $\delta\langle \psi_i \rangle(\vec{r})$ of these OP profiles. Some particular versions of such solitons have already been described in the literature. For example, in [33], a one-dimensional two-band superconductor with a simple interband interaction term was considered, whose ground state corresponded to $\langle \psi_1 \rangle$ and $\langle \psi_2 \rangle$ with zero relative phase. Then, a typical sine-Gordon soliton was constructed with the relative phase between the two OPs continuously changing from zero to 2π at $x = \pm\infty$, correspondingly. Similar solitons in the scalar sector of 2HDM were described in [34].

The existence of solitons in a given 2OP GL model depends on the geometry of the potential in the orbit space. For example, in order to support a one-dimensional soliton similar to the one described above, the Landau potential must have a certain "valley" (i.e., a region of low values of the potential) of nontrivial topology on the forward lightcone LC^+, that would include the global minimum and a saddle point. At $x \to -\infty$, $\langle \psi_i \rangle(x)$ approach their values at the global minimum. As x increases, the corresponding point $\langle r^\mu \rangle(x)$ in the orbit space moves away from the global minimum position, follows some path in the valley, and returns again to the global minimum. Small variations of $\langle \psi_i \rangle(x)$ would pull this path out of the valley, increasing its potential energy.

The existence, stability, and geometric properties (e.g., dimensionality) of these solitons are sensitive only to the general structure of the model, and do not require one to search for the explicit position of the extrema. Therefore, one can hope to obtain these criteria for a general 2OP GL model in terms of geometric constructions studied in this paper.

IX. MULTICOMPONENT ORDER PARAMETERS

So far, we assumed that the order parameters ψ_1 and ψ_2 are just complex numbers. However, in many physical situations one introduces *multicomponent order parameters*. Examples include 2HDM, superfluidity in ^{3}He, nonconventional superconductivity, spin-density waves, etc.

The formalism presented above is applicable to these cases as well. In fact, it was first developed in [28,29] specifically for 2HDM. Here, we discuss characteristic features that appear in a generic GL model with two N-vector order parameters and a U(N)-symmetric potential.

A. Modifications to the formalism

Let us assume that each ψ_i is an N-dimensional complex vector: $\psi_{i\alpha}$, $\alpha = 1, \ldots, N$. A U(N)-symmetric potential must depend on the order parameters only via scalar combinations $(\psi_i^\dagger \psi_j)$, $i,j,=1,2$, which parametrize the U(N) orbits. The only difference with the scalar case is that an additional term proportional to

$$(\psi_1^\dagger \psi_2)(\psi_2^\dagger \psi_1) \neq |\psi_1|^2 |\psi_2|^2 \tag{51}$$

appears in the potential, with a new independent coefficient b_3' in front. The definition of r^μ remains the same; however,

$$r^\mu r_\mu = 4[(\psi_1^\dagger \psi_2)(\psi_2^\dagger \psi_1) - |\psi_1|^2 |\psi_2|^2] \geqslant 0. \tag{52}$$

Therefore, the orbit space now is not only the surface, but also the *interior* of the forward lightcone LC^+. This removes the degree freedom in definition of $B_{\mu\nu}$, making it uniquely defined,

$$B^{\mu\nu} = \frac{1}{2}\begin{pmatrix} \frac{b_1+b_2}{2}+b_3 & -\mathrm{Re}(b_5+b_6) & \mathrm{Im}(b_5+b_6) & -\frac{b_1-b_2}{2} \\ -\mathrm{Re}(b_5+b_6) & b_3'+\mathrm{Re}\, b_4 & -\mathrm{Im}\, b_4 & \mathrm{Re}(b_5-b_6) \\ \mathrm{Im}(b_5+b_6) & -\mathrm{Im}\, b_4 & b_3'-\mathrm{Re}\, b_4 & -\mathrm{Im}(b_5-b_6) \\ -\frac{b_1-b_2}{2} & \mathrm{Re}(b_5-b_6) & -\mathrm{Im}(b_5-b_6) & \frac{b_1+b_2}{2}-b_3 \end{pmatrix}. \tag{53}$$

The requirement that the potential is stable in a strong sense implies that $B_{\mu\nu}$ must be positive definite *on and inside* LC^+. This leads not only to $B_0 > B_i$, but also to $B_0 > 0$. Note that due to the absence of freedom in $B_{\mu\nu}$, cases with singular $B_{\mu\nu}$ and with B_i of different signs must be considered as well.

B. Consequences

Let us discuss the modification of the above analysis due to the multicomponent order parameters.

A new phase appears, which is characterized by a stronger breaking of the symmetry of the potential. It corresponds to the global minimum of the potential $\langle r^\mu \rangle$ lying strictly inside the lightcone LC^+. This is possible only when $\langle \psi_1 \rangle$ and $\langle \psi_2 \rangle$ are not proportional to each other. In other words, one can always perform a simultaneous "intravector" $\mathrm{U}(N)$ transformation of both order parameters that makes them

$$\langle \psi_1 \rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_1 \end{pmatrix}, \quad \langle \psi_2 \rangle = \begin{pmatrix} 0 \\ \vdots \\ u \\ v_2 e^{i\xi} \end{pmatrix}, \tag{54}$$

where dots indicate zeros. Here, u, v_1, v_2, ξ are real, and u and v_1 must be nonzero in order for r^μ constructed from them to lie strictly inside LC^+.

Solution (54) with nonzero u preserves only a $\mathrm{U}(N-2)$ symmetry, while a normal solution lying on LC^+ and corresponding to $u=0$ preserves a $\mathrm{U}(N-1)$ symmetry. For example, in the context of the two-Higgs-doublet model ($N=2$), such a solution corresponds to a complete breaking of the electroweak symmetry group $\mathrm{SU}(2)\times\mathrm{U}(1)$. Such a phase breaks the electric charge conservation, and makes the photon massive.

Conditions when this phase appears were established in [28]. Since $\langle r^\mu \rangle$, which corresponds to such a nonsymmetric phase, is not restricted anymore to lie on the surface of LC^+, the extremum condition of the potential takes a very simple form,

$$B_{\mu\nu} r^\nu = A_\mu. \tag{55}$$

If $B_{\mu\nu}$ is nonsingular, then solution of Eq. (55) always exists and is unique. If the potential has any additional explicit symmetries, this symmetry is always conserved in this phase. If $B_{\mu\nu}$ is singular, then depending on A_μ, Eq. (55) can have an empty set or a continuum of solutions.

Whether the solution of Eq. (55) corresponds to a physically realizable extremum of the potential depends on whether $a_\mu = (B^{-1})_{\mu\nu} A^\nu$ lies inside LC^+. If it is so, then it can be a minimum or a saddle point. It is a minimum (and necessarily the global minimum) when $B_{\mu\nu}$ is positive definite in the entire Minkowski space, i.e., when all $B_i < 0$.

The search for the extrema on the forward lightcone LC^+ proceeds in the same way as before. One again introduces equipotential surfaces $\mathcal{M}^C$, but due to fixed eigenvalues B_i their geometry can be different. A typical $\mathcal{M}^C$ can now be any 3-quadric: a 3-hyperboloid, a 3-ellipsoid, a 3-cone, or a 3-paraboloid. The geometric reformulation of the problem remains unchanged: the search for the global minimum corresponds to the search for the unique 3-quadric with the base point a^μ that touches but never intersects the forward lightcone.

As a result, virtually all the statements about the number of extrema and minima, about the symmetries, and their spontaneous violation remain the same. The only difference is that r^μ can shift from the surface of LC^+ inwards, and in order for an extremum on LC^+ to be a minimum, this shift must also increase the potential. This means that the Lagrange multiplier λ in Eq. (16) must be positive. In fact, the eigenfrequencies ω^2 of oscillations that make N-vectors ψ_1 and ψ_2 nonparallel are proportional to λ. In 2HDM, they correspond to the masses of charged Higgs bosons [28,30].

X. CONCLUSIONS

The aim of this paper was to provide an exhaustive description of the general two-order-parameter model with all possible U(1)-symmetric quadratic and quartic interaction terms in the mean-field approximation. The principal difficulty in the study of this model lies in the fact that the Landau potential cannot be minimized with straightforward algebra. Here we showed that despite this difficulty, one can still learn a lot about the phase structure of this model. We developed the Minkowski-space formalism based on the reparametrization symmetry of the model and reformulated the minimization problem in simple geometric terms. We then proved several statements concerning the properties of the model (the number of extrema and minima, symmetries and their violation, and the phase diagram).

The most general 2OP GL model can be viewed as a "template" for many particular realizations of the two-order-parameter model used in various condensed-matter problems. We believe that by considering the most general case,

one can gain a more transparent understanding of phenomena taking place in particular situations, and one might even establish new links between seemingly unrelated models.

We also note that the general method used in this paper (consider the model in the most general case, find the group of reparametrization symmetries, and use it to find the structure behind the model) is very general and might turn out helpful in other circumstances.

There remain several directions for future work. First, using the dependence of the coefficients on temperature, pressure, etc., one can trace in detail the sequence of phase transitions as well as their critical properties in the mean-field approximation. Second, one should study modifications caused by the presence of external fields (e.g., magnetic fields for two-gap superconductors) and nontrivial boundary conditions. Third, one should analyze effects beyond the mean-field approximation; in particular, one should study how the symmetries of the model evolve under the renormalization-group flow. Fourth, one should closely examine the existence, stability, and dynamics of the solitons. Finally, an extension of the approach to models with several order parameters and/or with matric-valued OPs also appears to be feasible.

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APPENDIX A: MANIPULATION WITH 4-TENSOR $B_{\mu\nu}$

Here we collect some simple facts about the real symmetric 4-tensor $B_{\mu\nu}$. Let us first give explicit expressions for $B_{\mu\nu}$ with raised or lowered indices,

$$B^{\mu\nu} = \begin{pmatrix} B_{00} & B_{0j} \\ B_{0i} & B_{ij} \end{pmatrix}, \quad B^{\mu}{}_{\nu} = B_{\mu\alpha} g^{\alpha\nu} = \begin{pmatrix} B_{00} & -B_{0j} \\ B_{0i} & -B_{ij} \end{pmatrix},$$

$$B_{\mu\nu} = \begin{pmatrix} B_{00} & -B_{0j} \\ -B_{0i} & B_{ij} \end{pmatrix}. \tag{A1}$$

Here $i, j = 1,2,3$. Note that $B^{\mu}{}_{\nu}$ is not symmetric.

Upon an SO(3) rotation, B_{00} remains invariant, while B_{0i} and B_{ij} transform as a real 3-vector and a symmetric 3-tensor, respectively. Upon a boost with "rapidity" η, say, along the first axis, $B^{\mu\nu}$ transforms as

$$B^{\mu\nu} = \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{01} & b_{11} & b_{12} & b_{13} \\ b_{02} & b_{12} & b_{22} & b_{23} \\ b_{03} & b_{13} & b_{23} & b_{33} \end{pmatrix}$$

$$\to (B')^{\mu\nu} = \begin{pmatrix} b'_{00} & b'_{01} & b'_{02} & b'_{03} \\ b'_{01} & b'_{11} & b'_{12} & b'_{13} \\ b'_{02} & b'_{12} & b'_{22} & b'_{23} \\ b'_{03} & b'_{13} & b'_{23} & b'_{33} \end{pmatrix}, \tag{A2}$$

where

$$b'_{00} = \frac{b_{00} - b_{11}}{2} + \frac{b_{00} + b_{11}}{2} \cosh 2\eta + b_{01} \sinh 2\eta,$$

$$b'_{11} = -\frac{b_{00} - b_{11}}{2} + \frac{b_{00} + b_{11}}{2} \cosh 2\eta + b_{01} \sinh 2\eta,$$

$$b'_{01} = \frac{b_{00} + b_{11}}{2} \sinh 2\eta + b_{01} \cosh 2\eta,$$

$$b'_{0a} = b_{0a} \cosh \eta + b_{1a} \sinh \eta, \quad b'_{1a} = b_{1a} \cosh \eta + b_{0a} \sinh \eta,$$

$$b'_{ab} = b_{ab}, \quad a,b = 2,3.$$

The eigenvalues B_i and eigenvectors $e^{\mu}_{(i)}$ of $B_{\mu\nu}$ are defined according to

$$B_{\mu\nu} e^{\nu}_{(i)} = B_i g_{\mu\nu} e^{\nu}_{(i)}, \quad \text{or equivalently } B_{\mu}{}^{\nu} e_{(i)\nu} = B_i e_{(i)\mu}. \tag{A3}$$

The fact that $B_{\mu}{}^{\nu}$ is not symmetric implies that the some eigenvalues might be, in general, complex. However, as proven below, positive definiteness of $B_{\mu\nu}$ on the forward lightcone LC^+ ensures that they are real.

In the diagonal basis, one has

$$B_{\mu\nu} = \begin{pmatrix} B_0 & 0 & 0 & 0 \\ 0 & -B_1 & 0 & 0 \\ 0 & 0 & -B_2 & 0 \\ 0 & 0 & 0 & -B_3 \end{pmatrix},$$

$$B_{\mu}{}^{\nu} = \begin{pmatrix} B_0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_3 \end{pmatrix}.$$

If one considers a quadratic form in the space of 4-vectors p^{μ} constructed on $B_{\mu\nu}$, then in the diagonal basis it looks as

$$B_{\mu\nu} p^{\mu} p^{\nu} = B_0 p_0^2 - \sum_i B_i p_i^2.$$

This quadratic form is positive definite in the entire space of nonzero vectors p^{μ}, if and only if all B_i are *negative*.

APPENDIX B: POSITIVE DEFINITENESS OF V_4

A potential stable in a strong sense was defined as the one whose quartic part V_4 is strictly positive definite in the entire space of the order parameters ψ_i except the origin. In the orbit space it corresponds to $B_{\mu\nu} r^{\mu} r^{\nu}$ being positive definite on the entire forward lightcone LC^+ expect the apex. This criterion can be formulated in terms of the eigenvalues of $B^{\mu\nu}$.

Proposition 4. Tensor $B^{\mu\nu}$ is positive definite on the future lightcone except the apex if and only if the following conditions are met:

(i) $B^{\mu\nu}$ is diagonalizable by an SO(1,3) transformation.

(ii) All spacelike eigenvalues B_i are smaller than the timelike eigenvalue B_0.

Proof. Obviously, if $B^{\mu\nu}$ satisfies conditions (i) and (ii), then the positive definiteness follows immediately. So, one needs to prove that these conditions follow from the positive definiteness.

The first step is to prove that the positive definiteness on LC^+ implies that all the eigenvalues of $B^{\mu\nu}$ are real.

Suppose, on the contrary, that there exists a pair of nonzero complex eigenvalues, b and b^*, with respective (and necessarily complex) eigenvectors p^μ and q^μ,

$$B^\mu{}_\nu p^\nu = bp^\mu, \quad B^\mu{}_\nu q^\nu = b^* q^\mu.$$

One can show that there can be only one pair of complex eigenvalues, thus b is nondegenerate. Since $B^\mu{}_\nu$ is real, $q^\mu \propto p^{\mu *}$ (and can be taken equal to $p^{\mu *}$). These eigenvectors are orthogonal, $p^\mu q_\mu = 0$, which follows from the standard argument due to $b \neq b^*$, and can be normalized so that $p^\mu p_\mu = q^\mu q_\mu = 1$.

Consider now a nonzero real vector r^μ,

$$r^\mu = cp^\mu + c^* p^{*\mu},$$

such that $r^\mu r_\mu = c^2 + c^{*2} = 2|c|^2 \cos(2\phi_c) = 0$. At fixed $|c|$, four such vectors are possible. Take two of them: $r^\mu_{1/4}$ and $r^\mu_{3/4}$, corresponding to $\phi_c = \pi/4$ and $3\pi/4$. The quadratic form calculated on these vectors is

$$\begin{aligned} B_{\mu\nu} r^\mu r^\nu &= bc^2 + b^* c^{*2} = 2|b||c|^2 \cos(2\phi_c + \phi_b) \\ &= \mp 2|b||c|^2 \sin(\phi_b) \end{aligned}$$

for $r^\mu_{1/4}$ and $r^\mu_{3/4}$, respectively. Since b is not purely real, $\sin(\phi_b) \neq 0$; in one of the two cases $B_{\mu\nu} r^\mu r^\nu < 0$, which contradicts the assumption.

After all the eigenvalues of $B_{\mu\nu}$ are proven to be real, the eigenvectors can also be chosen all real and orthonormal. These eigenvectors cannot lie on LC^+ (otherwise there would be a flat direction of V_4), so there is one vector inside LC^+ with positive norm, norm $p^\mu_0 p_{0\mu} = 1$, and three spacelike eigenvectors with negative norms $p^\mu_i p_{i\mu} = -1$ for each $i = 1,2,3$. Thus, the transformation matrix T that diagonalizes $B^{\mu\nu}$ is real, and after diagonalization $B^{\mu\nu}$ takes the form $\mathrm{diag}(B_0, -B_1, -B_2, -B_3)$. Note that transformation T also conserves norm, so it can be realized as a transformation from the proper Lorentz group.

Now, the requirement that $B^{\mu\nu}$ is positive definite on LC^+ reads

$$B_0 - (B_1 \sin\theta\cos\phi + B_2 \sin\theta\sin\phi + B_3 \cos\theta) > 0$$

for all $0 \leqslant \theta \leqslant \pi$ and ϕ. This holds when B_0 is larger than any B_i. ■

Let us also see what changes for a potential stable in a weak sense. First, the statement that the eigenvalues are real and therefore eigenvectors can be also chosen real remains valid in this case. However, at least one eigenvector must now lie on the surface of LC^+. This means that $B^{\mu\nu}$ is in general not diagonalizable by the SO(1,3) transformation group. More details are given in Sec. IV.

APPENDIX C: NECESSARY CONDITION FOR THE SPONTANEOUS VIOLATION: EXPLICIT CALCULATIONS

Here, we show that the global minimum of the potential with all distinct B_i and $A_\mu = (A_0, A_1, A_2, 0)$ can spontaneously break the Z_2 symmetry given by reflections of the third axis, only if B_3 is the largest spacelike eigenvalue,

$$B_3 > B_1, B_2. \tag{C1}$$

We assume, of course, that the vector A_μ lies inside the caustic cone,

$$\frac{A_1^2}{(B_3 - B_1)^2} + \frac{A_2^2}{(B_3 - B_2)^2} < \frac{A_0^2}{(B_0 - B_3)^2}. \tag{C2}$$

his will be done by comparing the depth of the potential at the extrema that conserve and violate this symmetry. We will see that Eq. (C1) is necessary for the pair of symmetry-violating extrema to be the deepest ones.

If $\langle r^\mu \rangle$ is an extremum point, then the potential at this point is

$$V = -A_\mu \langle r^\mu \rangle + \frac{1}{2} B_{\mu\nu} \langle r^\mu \rangle \langle r^\nu \rangle = -\frac{1}{2} A_\mu \langle r^\mu \rangle = -\frac{1}{2} B_{\mu\nu} \langle r^\mu \rangle \langle r^\nu \rangle.$$

According to Eq. (18), the symmetry-violating extrema take place at

$$r_0 = r_0^{(3)} = \frac{A_0}{B_0 - B_3}.$$

The depth of the potential at this point is

$$|V_3| = \frac{1}{2}\left(\frac{A_0^2}{B_0 - B_3} + \frac{A_1^2}{B_3 - B_1} + \frac{A_2^2}{B_3 - B_2} \right).$$

Pick up another extremum (necessarily a symmetry-conserving one). It takes place at another r_0, which we rewrite as $r_0 \equiv r_0^{(3)} x$. The depth of the potential at this point is

$$\begin{aligned} |V| &= \frac{1}{2} r_0 (A_0 - A_1 n_1 - A_2 n_2) \\ &= \frac{1}{2} x \left[\frac{A_0^2}{B_0 - B_3} - \frac{A_1^2}{(B_0 - B_3) - (B_0 - B_1)x} \right. \\ &\quad \left. - \frac{A_2^2}{(B_0 - B_3) - (B_0 - B_2)x} \right]. \end{aligned}$$

Here, n_1, n_2 are

$$n_1 = \frac{A_1}{A_0 - (B_0 - B_1) r_0}, \quad n_2 = \frac{A_2}{A_0 - (B_0 - B_2) r_0}, \quad n_1^2 + n_2^2 = 1.$$

Note that the last equation, in fact, is the fourth-order equation for r_0.

The difference between the two depths can be presented, after some algebra, in the following way:

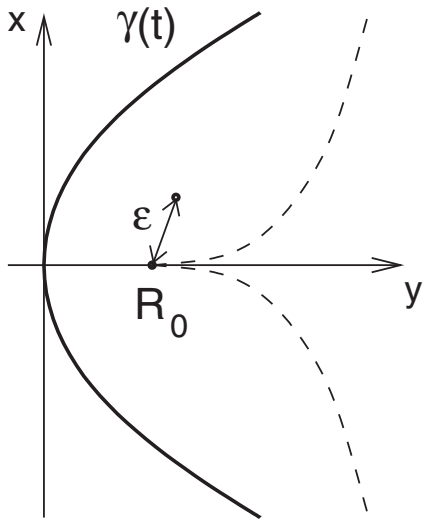

FIG. 7. Regular curve $\gamma(t)$ and its evolute (dashed line); see text.

$$|V_3| - |V| = \frac{(1-x)A_0}{2}\left(\frac{A_0}{B_0 - B_3} - \frac{A_1 n_1}{B_1 - B_3} - \frac{A_2 n_2}{B_2 - B_3}\right). \tag{C3}$$

The expression in large parentheses can be rewritten as $\alpha_\mu n^\mu$, where $n^\mu = (1, n_1, n_2, 0)$ and

$$\alpha^\mu = \left(\frac{A_0}{B_0 - B_3}, \frac{A_1}{B_1 - B_3}, \frac{A_2}{B_2 - B_3}, 0\right).$$

From the caustic condition (C2) one obtains $\alpha^\mu \alpha_\mu > 0$, i.e., the 4-vector α_μ lies strictly inside LC^+. On the other hand, n^μ lies on the surface of LC^+, and therefore $\alpha_\mu n^\mu > 0$. Thus, the sign of the depth difference is given solely by the value of x.

If B_3 is the largest spacelike eigenvalue, then all symmetry-conserving extrema correspond to $r_0 < r_0^{(3)}$, i.e., to $x < 1$. Therefore, all of them lie above the symmetry-violating points (and are saddle points, according to Proposition 2). If B_3 is not the largest spacelike eigenvalue, then there will necessarily be another extremum with $r_0 > r_0^{(3)}$, which corresponds to $x > 1$ and, therefore, lies deeper than the symmetry-violating points (which are saddle points in this case).

An alternative, somewhat longer way to prove condition (C1) using geometric properties of the potential was given in [29].

APPENDIX D: CRITICAL EXPONENT FOR THE DISTANCE SQUARED FUNCTION DEFINED ON A PLANAR CURVE

Let $\gamma(t)$ be a regular parametrization of a smooth plane curve; see, e.g., [35]. Take a point on this curve, assuming that it corresponds to $t=0$, and choose the coordinate frame at this point such that axis x is along the tangent and axis y is along the normal to the curve at this point; see Fig. 7. The curve can then be parametrized as $\gamma(t) = (X(t), Y(t))$, with

$$X(t) = t, \quad Y(t) = \frac{t^2}{2R_0} + a_n t^n + o(t^n). \tag{D1}$$

Here, R_0 is the curvature radius of γ at $t=0$, while $n > 2$ describes the next higher-order term.

Now, select a point $\vec{r} = (x, y)$ on this plane and calculate the distance squared from this point to the points of the curve, $\rho^2(t) \equiv [X(t) - x]^2 + [Y(t) - y]^2$. This function has points of extrema at some values of t. For a generic point $\vec{r}$, $\rho^2(t)$ will have a generic value at $t=0$. However, if $\vec{r}$ lies on the y axis, then $\rho^2(t)$ has a maximum or minimum at $t=0$. At a special ("critical") point along this axis, $\vec{r} = (0, R_0)$,

$$\rho^2(t) = X^2(t) + (Y(t) - R_0)^2 \approx \text{const} - 2R_0 a_n t^n + \frac{t^4}{4R_0^2}.$$

Therefore, the osculating circle has a threefold or fourfold contact with the curve γ, depending on whether $n=3$ or $n \geqslant 4$. As the point $\vec{r}$ moves along the y axis and passes through $(0, R_0)$, a bifurcation takes place of the function $\rho^2(t)$.

There are two possibilities to consider. If $\gamma(0)$ is a generic point on a generic smooth curve, then expansion (D1) starts from $n=3$, and at small t, $\rho^2(t)$ has a minimum and a maximum near $t=0$, both of which cannot be the global ones. Instead, if $\gamma(0)$ is an apex of the curve (this is the situation shown in Fig. 7), then expansion (D1) starts from $n=4$, and at this critical point the minimum of $\rho^2(t)$ splits into a minimum a maximum/minimum sequence. This is the only type of bifurcation in which the global minimum of the function $\rho^2(t)$ can participate, and we now focus on it.

Let us shift $\vec{r}$ away from the critical point by a small amount, $\vec{r} = (\varepsilon_x, R_0 + \varepsilon_y)$, and recalculate $\rho^2(t)$,

$$\rho^2(t) \approx \text{const} + t^4\left(\frac{1}{4R_0^2} - 2a_4 R_0\right) - 2\varepsilon_x t - \frac{\varepsilon_y t^2}{R_0}. \tag{D2}$$

For a generic curve $\gamma(t)$ (the second-order curves included), the coefficient in front of t^4 is nonzero. Finding the minimum of Eq. (D2), expanding ρ^2 near it, and extracting the coefficient in front of the quadratic term $(t - t_{\min})^2$, which should behave as $\propto |\varepsilon|^\delta$, gives us the value of δ. One can easily find that it depends on the *direction of approach* to the critical point,

$$\text{generic direction } (\varepsilon_x \neq 0) \to \delta = 2/3,$$

$$\text{symmetric approach } (\varepsilon_x = 0) \to \delta = 1. \tag{D3}$$

The latter case, by construction, corresponds effectively to the standard one-order-parameter Ginzburg-Landau model.

This study can be generalized to the $(n+1)$-dimensional case. Given a smooth n-manifold $\gamma(t_1, \ldots, t_n)$, one can choose a point of this manifold that would correspond to the global minimum, align the coordinate axes with the eigenvectors of the quadratic term, and parametrize the manifold as

$$X_1 = t_1, \ldots, X_n = t_n, \quad Y \approx \frac{t_1^2}{2R_1} + \cdots + \frac{t_n^2}{2R_n} + a_4 t_1^4. \tag{D4}$$

Here, we labeled the axes according to $R_1 < R_2 < \cdots < R_n$; the bifurcation we study is at $\vec{r} = (0, \ldots, 0, R_1)$. The calculations can be repeated giving the same result: if $\varepsilon_x \neq 0$, then $\delta = 2/3$, otherwise $\delta = 1$. Now ε must be understood as the distance from the closest among the cusp points of the evolute. This result does not depend on the particular shape of the manifold, since it is essentially driven by the fourth-order nature of the bifurcation point.

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